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Bounds for Distributions with Monotone Hazard Rate

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1. INTRODUCTION.

If F is a probability distribution such that $F(0-) = 0$ and $\int_{0-}^{\infty} x^r dF(x) = \mu_r < \infty$, and if $r,t > 0$, then according to Markov's inequality,

$$0 \leq 1 - F(t-) \leq \begin{cases} \mu^r/t^r, & t \geq \mu_r \\ 1, & t \leq \mu_r. \end{cases} \tag{1.1}$$

This inequality is known to be sharp; indeed, for each positive r and t there exist distributions satisfying the conditions of (1.1) and attaining equality.

A number of improvements of (1.1) have been obtained under additional assumptions about the distribution F. Perhaps the most notable of these is the result of Gauss (1821) which applies in case $1 - F(x)$ is convex in $x \geq 0$, and predates any version of (1.1). Hypotheses similar to that of Gauss have been used by a number of authors to obtain improvements; much of this work has been summarized by Fréchet (1950). Improvements of the classical bounds were studied by Mallows (1956) under restrictions on the number of sign changes of some derivative of the distribution, and also with restrictions on the size of the derivative. This work extends the result of Gauss as well as that of Markov (1898) which utilized bounds on the density. Recently, Mallows (1962) has extended his earlier work, as well as the results and methods of Krein (1951), to obtain inequalities on distributions having n specified moments and whose first s derivatives satisfy certain boundedness and sign change conditions.

In this paper, we obtain sharp upper and lower bounds for $1 - F(t-)$ under a variety of conditions, particularly that the hazard rate is monotone. These conditions are of interest for two reasons: First, they are sufficient to yield quite striking improvements of (1.1), and second, they are natural to many situations in life testing, reliability, actuarial science, and other areas of statistical interest.

A distribution F is said to have increasing (decreasing) hazard rate, denoted by IHR(DHR), if $\log[1 - F(x)]$ is concave where finite (convex on $[0,\infty)$). If F has a density f, then the ratio

$$q(x) = f(x)/[1 - F(x)]$$

is defined for $F(x) < 1$, and is called the <u>hazard rate</u>. It is easily seen that $\log[1 - F(x)]$ is concave (convex) in $x \geq 0$ if and only if $q(x)$ is increasing (decreasing) in $x \geq 0$.

The practical interest of the hazard rate derives from its probabilistic interpretation: If F is a life distribution, then $q(x)dx$ may be regarded as the conditional probability of death in $(x, x + dx)$ given survival to age x.

The property of monotone hazard rate is connected with the theory of total positivity in the following way. A distribution F is IHR if and only if $1 - F(x - y)$ is totally positive of order 2 in real x and y (see Schoenberg (1951) for a definition of terms). A distribution F is DHR if and only if $1 - F(x + y)$ is totally positive of order 2 in $x + y \geq 0$. Properties of distributions with monotone hazard rate have been investigated by Barlow, Marshall and Proschan (1963).

We pay particular attention to the question of sharpness of the inequalities given, and to the conditions for equality. Examples attaining equality serve not only to prove sharpness, but also indicate what stronger assumptions may yield a further improvement of the inequality. For if a property is enjoyed by a distribution attaining equality, then the assumption of that property cannot result in further improvement. Where uniqueness of a distribution attaining equality can be shown, then of course strict inequality holds in all other cases.

The statement of (1.1) for $r > 0$ is in reality no more general than its statement for $r = 1$. This is because of the fact that for $r = 1$, (1.1) may be written in the form

$$P\{X \geq t\} \leq \mu/t$$

where $\mu = E(X)$. With $X = Y^r$, one then obtains (1.1) for arbitrary $r > 0$. The results of this paper cannot be so simply extended, because the property of monotone hazard rate need not be preserved under a transformation of the form $X = Y^r$.

Throughout this paper we assume unless otherwise stated that distributions are right continuous.

2. METHODS OF PROOF.

If X is a random variable satisfying $P\{X \epsilon I\} = 1$ and certain moments of X are known, there is a standard method for obtaining a sharp upper bound for the probability that X lies in some specified set $\mathcal{J} \subset I$. If $\mathcal{a}$ is the class of polynomials $h(x) = \Sigma a_j x^j$ where (i) $a_j = 0$ unless the j^{th} moment of X is known, and (ii) $h(x)$ dominates the characteristic (indicator) function of $\mathcal{J}$ on I, then

$$P\{X \epsilon \mathcal{J}\} \leq \inf_{\mathcal{a}} Eh(X) \tag{2.1}$$

(see Marshall and Olkin (1961) for a more general discussion). The usual proof of Markov's inequality (1.1) is of this form where the minimizing polynomial is x^r/t^r.

This proof of Markov's inequality does not seem adaptable to the case in which other kinds of information are available about the distribution F. We consider an alternate proof based upon the following lemma: If ζ is an increasing function on $[0,\infty)$ and G_1, G_2 are probability distributions satisfying $G_1(x) \leq G_2(x)$ for all x, then

$$\int_{0-}^{\infty} \zeta(x) dG_1(x) \geq \int_{0-}^{\infty} \zeta(x) dG_2(x). \tag{2.2}$$

To apply this, observe that

$$1 - F(x) \geq 1 - G(x) = \begin{cases} 1, & x < 0 \\ 1 - F(t-), & 0 \leq x < t \\ x \geq t. & \end{cases} \tag{2.3}$$

Then since x^r/t^r is increasing in x,

$$\frac{\mu_r}{t^r} = \int_{0-}^{\infty} \frac{x^r}{t^r} dF(x) \geq \int_{0-}^{\infty} \frac{x^r}{t^r} dG(x) = 1 - F(t-).$$

Some kinds of information about F readily yield a sharpening of (2.3) with consequent improvement of (1.1), and we illustrate with two simple examples.

Example 2.1. If $F(x)$ is convex in $(0,t)$, then

$$1 - F(x) \geq \begin{cases} 1 - xF(t-)/t, & x \leq t \\ 0, & x > t. \end{cases}$$

Using this, one obtains

$$1 - F(t-) \leq \frac{\mu_r}{t^r} - \frac{1}{r}\left(1 - \frac{\mu_r}{t^r}\right), \tag{2.4}$$

an improvement of (1.1) due to Narumi (1923).

Example 2.2. If $1 - F(x)$ is convex, $x \geq 0$ (e.g., if F is the distribution of a random variable $X = |Y|$ where Y has a density with unique mode at 0), then $1 - F(x)$ has a supporting line at $t > 0$, so that there exists $\alpha \leq 1$ such that

$$1 - F(x) \geq \begin{cases} 1, & x < 0 \\ \alpha + [1 - F(x) - \alpha]x/t, & 0 \leq x < \alpha t/[\alpha - 1 + F(t)] \\ 0, & x \geq \alpha t/[\alpha - 1 + F(t)]. \end{cases} \tag{2.5}$$

Thus for some $\alpha \leq 1$, $\mu_r \geq (\alpha t)^{r+1}/(r + 1)t(\alpha - 1 + F(t))^r$, or

$$1 - F(t) \leq \alpha - \alpha^{1+1/r} t/(r + 1)^{1/r} \mu_r^{1/r} \equiv \varphi(\alpha). \tag{2.6}$$

Though we have no way of obtaining α to satisfy (2.5), we do obtain a valid bound by maximizing $\varphi(\alpha)$ for $\alpha \leq 1$. This maximum occurs at $\alpha = r\mu_r^{1/r}/t(r+1)^{1-1/r}$ if $t \geq r\mu_r^{1/r}(r+1)^{1-1/r} = t_0$, and at $\alpha = 1$ if $t \leq t_0$. Thus

$$1 - F(t) \leq \begin{cases} 1 - t/(r+1)^{1/r}\mu_r^{1/r}, & t \leq t_0 \\ [\mu_r/t^r][r/(r+1)]^r & t \geq t_0. \end{cases} \tag{2.7}$$

This result was obtained by Camp (1922) and Meidell (1922). For $r = 2$, it is essentially equivalent to Gauss' result of 1821, and the method of the above proof is due to Gauss.

The method of Example 2.2 has the disadvantage of providing no inequality unless the problem of maximizing φ can be solved; this is in contrast to the method utilizing (2.1), where a valid bound is provided by any h satisfying (i) and (ii).

We use a third method in Sections 4 and 5, which may be described as follows. Let $\mathcal{F}$ be a family of probability distributions. Call $\mathcal{B} \subset \mathcal{F}$ extremal for $\mathcal{F}$ on T if for each $t \varepsilon T$ and $F \varepsilon \mathcal{F}$, there exists $G \varepsilon \mathcal{B}$ such that $F(t) = G(t)$. If $\mathcal{B}$ is extremal for $\mathcal{F}$ and $F \varepsilon \mathcal{F}$, then clearly

$$\inf_{\mathcal{B}} G(t) \leq F(t) \leq \sup_{\mathcal{B}} G(t).$$

If the family $\mathcal{B}$ is sufficiently small, the bound may not be difficult to obtain. This method has been used by Royden (1953) and Mallows (1956).

Our proofs that $\mathcal{G}$ is extremal involve a parameterization of $\mathcal{G}$: $\mathcal{G} = \{G_\alpha : \alpha \varepsilon I\}$. We single out a crossing of F and G_α, and show that this crossing must occur at each $t \varepsilon T$ as α ranges over I. Although this is conceptually simple, it is usually difficult to rigorize.

Example 2.3. Let $\mathcal{F}$ be the class of distributions F where F is convex on its interval of support and satisfies $F(0) = 0$, $\int_{0-}^{\infty} x dF(x) = \mu_1$. Let $\mathcal{G} = \{G_\alpha : 0 \leq \alpha \leq \mu_1\}$, where

$$G_\alpha(x) = \begin{cases} 0, & x < \alpha \\ \dfrac{x - \alpha}{2(\mu_1 - \alpha)}, & \alpha \leq x < 2\mu_1 - \alpha \\ 1, & x \geq 2\mu_1 - \alpha. \end{cases}$$

Suppose that $F \varepsilon \mathcal{F}$. Then F and G have at most two crossings. It is not difficult to see graphically (we make no attempt at a rigorous proof) that the first crossing must range over the interval $T = [0, \mu_1]$ as α ranges over the same interval. For $0 < t \leq \mu_1$, we compute

$$\sup_{0 \leq \alpha \leq \mu_1} G_\alpha(t) = G_0(t) = t/2\mu_1$$

and conclude that

$$F(t) \leq t/2\mu_1, \quad t \leq \mu_1. \tag{2.8}$$

We mention two other useful methods. Inequality (3.8) can be obtained by an application of Jensen's inequlaity, as can (2.7) in case $t \leq t_0$. Finally, we give another proof of (1.1) which, suitably modified, yields a simple proof of (3.10). The distribution G defined by

$$1 - G(x) = \begin{cases} 1, & x < 0 \\ \mu_r/t^r, & 0 \leq x < t \\ 0, & x \geq t \end{cases}$$

has r^{th} moment $\mu_r = \int_{0-}^{\infty} x^r dF(x)$. Hence F and G must cross at least once; such a crossing can occur only in the interval $(0,t)$, and thus

$$1 - F(t-) \leq 1 - G(t-) = \mu_r/t_r.$$

The ideas of this proof are also useful in Section 4, where more than one moment is known.

3. BOUNDS FOR $1 - F$ WHEN F HAS MONOTONE HAZARD RATE.

We introduce this section with some general lemmas that are later applied to obtain more specific results.

Let $t > 0$,

$$1 - G_{z;w}(x) = \begin{cases} 1, & x \le z , \\ w^{\frac{x-z}{t-z}}, & x \ge z , \end{cases}$$

and let

$$1 - G_z(x) = 1 - G_{z;1-F(t-)}(x) .$$

<u>Lemma 3.1</u>. Let F be IHR, $F(0) = 0$. Let ζ be a function strictly increasing on $[0,\infty)$ such that $\int_0^\infty \zeta(x)dF(x) = \nu$ exists finitely. Then

$$\psi(w) = \sup_{0 \le z \le t} \int_0^\infty \zeta(x)dG_{z;w}(x)$$

is strictly increasing, and if $\psi(1 - F(t -)) < \infty$,

$$1 - F(t -) \ge \begin{cases} \psi^{-1}(\nu), & \zeta(t) \le \nu \\ 0 , & \zeta(t) > \nu , \end{cases} \qquad (3.1)$$

where $\psi^{-1}(\nu) = \sup \{w : \psi(w) \le \nu \}$.

<u>Proof</u>. Note that $G_{z;w}(x)$ is decreasing in w for fixed x and z. Since ζ is strictly increasing, this means $\int_0^\infty \zeta(x)dG_{z;w}(x)$ is strictly increasing in w, so $\psi(w)$ is strictly increasing. Since $\psi(0) = \zeta(t)$, $\psi^{-1}(\nu)$ is defined when $\zeta(t) \le \nu$.

Since $\log[1 - F(x)]$ is concave, there exists z_0, $0 \leq z_0 \leq t$ such that $F(x) \geq G_{z_0}(x)$ for all x. Since ζ is increasing,

$$\nu = \int_0^\infty \zeta(x)dF(x) \leq \int_0^\infty \zeta(x)dG_0(x) \leq \sup_{0\leq z\leq t} \int_0^\infty \zeta(x)dG_z(x) = \psi(1 - F(t -)) \qquad (3.2)$$

and (3.1) follows.||

Note that no use was made of the condition $F(0) = 0$ other than to confine z_0 to $[0,t]$ rather than $(-\infty, t]$.

Lemma 3.1'. If $\zeta(t) \leq \nu$ and ψ is continuous at ν, equality is attained in (3.1) uniquely by the distribution $G_{z^*;\psi^{-1}(\nu)}(x)$, where z^* is defined by $\int_0^\infty \zeta(x)dG_{z^*;\psi^{-1}(\nu)}(x) = \sup_{0\leq z\leq t} \int_0^\infty \zeta(x)dG_{z;\psi^{-1}(\nu)}(x)$.

If $\zeta(t) > \nu$ and $\zeta(s) = \nu$ has a solution, then, e.g., the distribution degenerate at s achieves equality.

Proof. If $\zeta(t) \leq \nu$, then $\psi^{-1}(\nu)$ exists. Since ψ is continuous at ν,

$$\nu = \psi(\psi^{-1}(\nu)) = \sup_{0\leq z\leq t} \int_0^\infty \zeta(x)dG_{z;\psi^{-1}(\nu)}(x) = \int_0^\infty \zeta(x)dG_{z^*;\psi^{-1}(\nu)}(x) =$$

$$= \psi[1 - G_{z^*;\psi^{-1}(\nu)}(t -)],$$

so that the hypotheses of Lemma 3.1 are satisfied when $F = G_{z^*;\psi^{-1}(\nu)}$ and equality is attained. Uniqueness follows from the fact that equality must hold in (3.2) if it holds in (3.1).||

Lemma 3.2. If the conditions of Lemma 3.1 are satisfied and if, in addition, ζ is convex, then $\psi(1 - F(t -)) = \int_0^\infty \zeta(x)dG_0(x)$ whenever $\zeta(t) \le \nu$.

Proof. $\int_0^\infty \zeta(x)dG_z(x) = -\frac{L}{t - z} \int_z^\infty \zeta(x)\exp(\frac{x - z}{t - z}L)dx =$

$$= -L\int_0^\infty \zeta(z(1 - y) + ty)e^{yL}dy \equiv \varphi(z),$$

where $L = \log[1 - F(t-)]$. Since ζ is convex, φ is also convex, and $\sup_{0\le z\le t} \varphi(z) = \varphi(0)$ or $\varphi(t)$. If $\zeta(t) < \nu$, then $\sup_{0\le z\le t} \varphi(z) = \varphi(t)$ implies by (3.2) that $\psi(1 - F(t -)) = \varphi(t) = \zeta(t) \ge \nu$, a contradiction, so that $\psi(1 - F(t -)) = \sup_{0\le z\le t} \varphi(z) = \varphi(0) = \int_0^\infty \zeta(x)dG_0(x)$. If $\zeta(t) = \nu$, the result follows by limiting arguments. ||

From (3.2) and Lemma 3.2, it follows that if ζ is strictly increasing and convex on $[0,\infty)$, and if $\zeta(t) \le \nu$, then

$$\nu = \int_0^\infty \zeta(x)dF(x) \le \int_0^\infty \zeta(x)\omega e^{-\omega x}dx$$

where $\omega = -t^{-1}\log(1 - F(t -))$. This inequality is to be compared with the inequality

$$\int_0^\infty \zeta(x)dF(x) \le \int_0^\infty \zeta(x)\cdot \frac{1}{\mu_1} e^{-x/\mu_1}dx \tag{3.3}$$

where $\mu_1 = \int_0^\infty xdF(x)$ and ζ need only be convex. Inequality (3.3) follows from an integration by parts and the fact (Karlin, Proschan and Barlow, 1961) that $1 - F(x)$ crosses e^{-x/μ_1} exactly once, the crossing being from above. Inequality (3.3) is due to Karlin and Novikoff (1962). ●

Lemma 3.3. Let F be IHR, $F(0) = 0$. Let ζ be a function strictly decreasing on $[0,\infty)$ such that $\int_0^\infty \zeta(x)dF(x) = \nu$ exists finitely. Then $\nu \geq \inf_{0\leq z\leq t} \int_0^\infty \zeta(x)dG_z(x)$,

$$\psi(w) = \inf_{0\leq z\leq t} \int_0^\infty \zeta(x)dG_{z;w}(x)$$

is strictly decreasing, and

$$1 - F(t -) \geq \begin{cases} \psi^{-1}(\nu), & \zeta(t) \geq \nu \\ 0\ , & \zeta(t) < \nu\ , \end{cases} \tag{3.4}$$

where $\psi^{-1}(\nu) = \inf\,\{w:\psi(w) \leq \nu\}$.

The proof of Lemma 3.3 is essentially the same as the proof of Lemma 3.1, and will be omitted. The obvious analogs of Lemma 3.1' and Lemma 3.2 (with concavity replacing convexity) are also omitted.

Let $1 - H_a(x) = \begin{cases} e^{-ax}, & x < t \\ 0, & x \geq t\ . \end{cases}$

Lemma 3.4. Let F be IHR, $F(0) = 0$, and let ζ be a function strictly increasing on $[0,\infty)$ such that $\int_0^\infty \zeta(x)dF(x) = \nu$ exists finitely. Then

$$\nu = \int_0^\infty \zeta(x)dH_a(x)$$

has a solution a_0 if and only if $\nu \leq \zeta(t)$; in this case, a_0 is unique, and

$$1 - F(t) \leq \begin{cases} 1 & \nu > \zeta(t) \\ e^{-a_0 t} & \nu \leq \zeta(t). \end{cases} \tag{3.5}$$

Proof. There is at most one crossing of $1 - H_a(x)$ by $1 - F(x)$ in $(0,t)$, and if such a crossing exists, it is from above (Karlin, Proschan, Barlow, 1961). If a_0 exists, $\nu = \int_0^\infty \zeta(x)dF(x) = \int_0^\infty \zeta(x)dH_{a_0}(x)$ and ζ strictly increasing implies F and H_{a_0} are not stochastically ordered. Thus $1 - F(t) \leq e^{-a_0 t}$.

If a_0 exists, then $\nu = \int_0^\infty \zeta(x)dH_{a_0}(x) \leq \int_0^\infty \zeta(x)dH_0(x) = \zeta(t)$; if $\zeta(t) \geq \nu$, then $\zeta(t) = \int_0^\infty \zeta(x)dH_0(x) \geq \nu \geq \zeta(0) = \lim_{a\to\infty} \int_0^\infty \zeta(x)dH_a(x)$ together with continuity of $\int_0^\infty \zeta(x)dH_a(x)$ implies a_0 exists. Uniqueness of a_0 follows from the stochastic ordering of the H_a and monotonicity of ζ. ||

Remark. Examination of the above proof shows that (3.5) still holds if the hypothesis that F is IHR is replaced by the weaker condition that $x^{-1}\log[1 - F(x)]$ is decreasing in $x \leq t$.

Lemma 3.4'. If a_0 exists, then equality in (3.5) is uniquely attained by H_{a_0}. If a_0 does not exist and ζ is continuous, then $\zeta(s) = \nu$ has a solution $s_0 > t$ and the distribution degenerate at s_0 attains equality.

Proof. We need only prove uniqueness when a_0 exists. Since $\log[1 - F(x)]$ is concave, $1 - F(t) = 1 - H_{a_0}(t)$ implies $1 - F(x) \geq 1 - H_{a_0}(x)$ for all x in $[0,t]$, and hence for all x. This together with $\nu = \int_0^\infty \zeta(x)dH_{a_0}(x) = \int_0^\infty \zeta(x)dF(x)$ implies $1 - F(x) =$ $= 1 - H_{a_0}(x)$ for all x. ||

Remark. In case $\nu \leq \zeta(t)$, the distribution H_{a_0} attaining equality

is not right continuous. If right continuity is demanded, equality cannot be attained in (3.5), but the bound can be approximated by a distribution of the form

$$1 - H(x) = \begin{cases} e^{-a_\varepsilon x}, & x < t + \varepsilon \\ 0, & x \geq t + \varepsilon, \end{cases}$$

where a_ε is determined by $\int_0^\infty \zeta(x)dH(x) = \nu$.

Right continuity of F was not used in the proof of (3.5), and hence $F(t)$ can be replaced by $F(t-)$ in (3.5). Of course, the right continuous version of H_{a_0} attains equality in (3.5) so modified.

The analog of Lemma 3.4 for decreasing ζ is straightforward, and is omitted.

Let $t > 0$,

$$1 - K_{\alpha;w}(x) = \begin{cases} \alpha(w/\alpha)^{x/t}, & 0 \leq w < \alpha \leq 1, \quad x \geq 0, \\ 1, & x < 0, \end{cases}$$

and let

$$1 - K_\alpha(x) = 1 - K_{\alpha,1-F(t)}(x).$$

<u>Lemma 3.5</u>. Let F be DHR, $F(0-) = 0$. Let ζ be a function strictly increasing on $[0,\infty)$ such that $\int_{0-}^\infty \zeta(x)dF(x) = \nu$ exists finitely. Then

$$\psi(w) = \inf_{1 \geq \alpha > w} \int_{0-}^\infty \zeta(x)dK_{\alpha;w}(x)$$

is strictly increasing, and

$$1 - F(t) \leq \psi^{-1}(\nu), \tag{3.6}$$

where $\psi^{-1}(\nu) = \inf \{w : \psi(w) \geq \nu\} < 1.$

Proof. Since $\log[1 - F(x)]$ is convex and $t > 0$, there exists $\alpha_0 < 1$ such that $1 - K_{\alpha_0}(x) \leq 1 - F(x)$ for all x. Since ζ is increasing,

$$\nu = \int_{0-}^{\infty} \zeta(x)dF(x) \geq \int_{0-}^{\infty} \zeta(x)dK_{\alpha_0}(x) \geq \inf_{1 \geq \alpha > 1-F(t)} \int_{0-}^{\infty} \zeta(x)dK_{\alpha}(x) = \psi(1 - F(t)).$$

As in the proof of Lemma 3.1, $\psi(w)$ is strictly increasing in w, so that (3.6) follows if $\psi^{-1}(\nu)$ is defined. But $\{w:\psi(w) \geq \nu\}$ is not empty, since

$$\lim_{w \uparrow 1} \psi(w) = \lim_{a \downarrow 0} \int_{0}^{\infty} \zeta(x)ae^{-ax}dx = \lim_{M \to \infty} \lim_{a \downarrow 0} \int_{M}^{\infty} \zeta(x)ae^{-ax}dx \geq \lim_{M \to \infty} \zeta(M) > \nu .$$

Since $\lim_{w \to 1} \psi(w) > \nu$, there exists $w < 1$ satisfying $\psi(w) > \nu$. This implies $\psi^{-1}(\nu) < 1.$ ||

Lemma 3.5'. Equality is attained in (3.6) uniquely by the distribution $K_{\alpha^*, \psi^{-1}(\nu)}(x)$, where α^* is defined by $\int_{0}^{\infty} \zeta(x)dK_{\alpha^*, \psi^{-1}(\nu)}(x) =$

$$= \inf_{\alpha \geq \psi^{-1}(\nu)} \int_{0}^{\infty} \zeta(x)dK_{\alpha, \psi^{-1}(\nu)}(x).$$

The proof of this is similar to the proof of Lemma 3.1'. We omit the analog of Lemma 3.5 for decreasing ζ; its statement is obtained by substituting the words "decreasing" for "increasing" and "supremum" for

"infimum" in the statement of Lemma 3.5. The direction of inequality (3.6) is then unchanged.

Lemma 3.6. Let F be DHR, $F(0-) = 0$, and let ζ be a strictly decreasing positive function on $[0,\infty)$ such that $\int_{0-}^{\infty} \zeta(x)dF(x) = \nu$ exists finitely. Then

$$\psi(w) = \int_0^t \frac{-\log w}{t} w^{\frac{x}{t}} \zeta(x)\, dx$$

is continuous and strictly increasing in $w \epsilon [0,1]$, and

$$1 - F(t) \geq \psi^{-1}(\nu) > 0 \ . \qquad (3.7)$$

Proof. Since ζ is positive and $\log[1 - F(x)]$ is convex,

$$\nu = \int_0^{\infty} \zeta(x)dF(x) \geq \int_0^t \zeta(x)dF(x) \geq \int_0^t \zeta(x)d\left(1 - [1 - F(t)]^{\frac{x}{t}}\right) = \psi(1 - F(t)).$$

One concludes that ψ is continuous and strictly decreasing, that $\lim_{w\to 0} \psi(w) = \zeta(0) > \nu$ and that $\lim_{w\to 1} \psi(w) = 0$ by considering the integrand in the definition of ψ. Thus $\psi^{-1}(\nu)$ exists. Since $\nu > 0$, it follows that $\psi^{-1}(\nu) > 0$. ||

Lemma 3.6'. Equality is attained in (3.7) uniquely by the (improper) distribution

$$1 - G(x) = \begin{cases} [\psi^{-1}(\nu)]^{\frac{x}{t}} , & 0 \leq x \leq t \\ \psi^{-1}(\nu) , & x \geq t. \end{cases}$$

If $\lim_{m\to\infty} \int_m^\infty \zeta(x)ae^{-ax}dx = 0$ <u>uniformly</u> in a, $0 < a < \delta$ for some $\delta > 0$, then for sufficiently small $\epsilon > 0$, there exists a proper distribution satisfying the conditions of Lemma 3.6 with the value $1 - \psi^{-1}(\nu) - \epsilon$ at t, so that no sharpening of (3.7) is possible.

Before proving this result, we note that $\lim_{x\to\infty} \zeta(x) = 0$ implies $\lim_{m\to\infty} \int_m^\infty \zeta(x)ae^{-ax}dx \le \lim_{m\to\infty} \zeta(m) = 0$, so that the limit is uniform in a.

<u>Proof</u>. Choose ϵ so small that $\int_{0-}^\infty \zeta(x)d\{1 - [\psi^{-1}(\nu) + \epsilon]^{x/t}\} > \nu$, possible since $\lim_{\epsilon\to 0} \int_{0-}^{t} \zeta(x)d\{1 - [\psi^{-1}(\nu) + \epsilon]^{x/t}\} = \int_{0-}^{t} \zeta(x)d\{1 - [\psi^{-1}(\nu)]^{x/t}\} = \nu$, and since $\zeta(x) > 0$ for all x. Choose a_0 to satisfy $\int_0^\infty \zeta(x)dG_a(x) = \nu$, where

$$1 - G_a(x) = \begin{cases} [\psi^{-1}(\nu) + \epsilon]^{x/t}, & 0 \le x \le t, \\ [\psi^{-1}(\nu) + \epsilon]e^{-a(x-t)}, & x \ge t. \end{cases}$$

In order to show that a_0 exists, note first that by choice of ϵ, $\int_{0-}^\infty \zeta(x)dG_a(x) > \nu$ when $a = -t^{-1}\log[\psi^{-1}(\nu) + \epsilon]$. Then since $\zeta(x)$ is uniformly integrable with respect to G_a, $a < \delta$, $\lim_{a\to 0} \int_{0-}^\infty \zeta(x)dG_a(x) = \int_{0-}^\infty \zeta(x)dG_0(x) < \nu$ [Loève, P.183,(1960)]. By continuity of $\int_{0-}^\infty \zeta(x)dG_a(x)$, a_0 exists. Since $a_0 < -t^{-1}\log[\psi^{-1}(\nu) + \epsilon]$, G_{a_0} is DHR.||

We do not give an analog to Lemma 3.6 for $\zeta(t)$ increasing; instead we prove

<u>Lemma 3.7</u>. Let F be DHR, $F(0-) = 0$. If $\zeta(x)$ is an increasing function on $[0,\infty)$ such that $\lim_{x\to\infty} \zeta(x) = \infty$ and such that $\int_{0-}^{\infty} \zeta(x)dF(x) = \nu < \infty$, then the inequality $1 - F(t) \geq 0$ is sharp for all $t > 0$. That is, no non-trivial lower bound can be given.

<u>Proof</u>. Since $\nu < \infty$, $\int_0^{\infty} \zeta(x)be^{-bx}dx < \infty$ for all $b > b_0 =$ $= \lim_{x\to\infty} F'(x)/[1 - F(x)]$, where $F'(x) = dF(x)/dx$. Let $a = [\nu - \zeta(0)]/[\int_0^{\infty} \zeta(x)be^{-bx}dx - \zeta(0)]$. Then $\lim_{b\downarrow b_0} a =$ $= [\nu - \zeta(0)]/[\lim_{x\to\infty} \zeta(x) - \zeta(0)] = 0$, so that for $b - b_0$ sufficiently small,

$$1 - G_b(x) = \begin{cases} 1, & x < 0, \\ ae^{-bx}, & x \geq 0 \end{cases}$$

is a distribution function satisfying the conditions of the lemma. But $\lim_{b\to b_0} 1 - G_b(t) = 0$. ||

Lemma 3.7 is still true even when a density is required to exist, as can be seen by considering distributions of the form

$$1 - G(x) = \begin{cases} e^{-\alpha x}, & 0 \leq x < t \\ e^{-\beta x-(\alpha-\beta)t}, & x \geq t, \end{cases}$$

where $\int_0^{\infty} \zeta(x)\alpha e^{-\alpha x}dx \leq \nu$ and β is determined by $\int_0^{\infty} \zeta(x)dG(x) = \nu$.

3.1 BOUNDS FOR $1 - F$, r^{th} MOMENT GIVEN.

Theorem 3.8. If F is IHR, $F(0) = 0$, $r \geq 1$ and $\int_0^\infty x^r dF(x) = \mu_r$, then

$$1 - F(t-) \geq \begin{cases} \exp[-t/\lambda_r^{1/r}], & t \leq \mu_r^{1/r} \\ 0, & t > \mu_r^{1/r}, \end{cases} \tag{3.8}$$

where $\lambda_r = \mu_r/\Gamma(r+1)$. This inequality is sharp.

Proof. This theorem is an immediate application of Lemmas 3.1, 3.1' and 3.2, where $\zeta(x) = x^r$. ||

In case $r = 1$ and F is continuous, (3.8) has an elegant direct proof. Since $\log[1 - F(x)]$ is concave, it follows from Jensen's inequality that

$$\log[1 - F(\lambda_1)] \geq \int_0^\infty \log[1 - F(x)]dF(x) = \int_0^1 \log(1-u)du = -1,$$

hence $1 - F(\lambda_1) \geq e^{-1}$. Since $[1 - F(t)]^{1/t} \geq [1 - F(\lambda_1)]^{1/\lambda_1}$ for $t \leq \lambda_1$ (See Barlow, Proschan and Marshall, 1963), we have

$$1 - F(t) \geq e^{-t/\lambda_1}. \; ||$$

The above proof can be easily modified with limiting arguments to include the case that F is not continuous. S.Karlin has pointed out that this proof can also be generalized to include the cases $r > 1$.

Theorem 3.9. Let F be IHR, $F(0) = 0$, $r > 0$, and $\int_0^\infty x^r dF(x) = \mu_r$. Then

$$\mu_r = rt^r \int_0^1 x^{r-1} w^x dx \tag{3.9}$$

has a solution w_0 if and only if $t \geq \mu_r^{1/r}$. In this case, w_0 is unique, and

$$1 - F(t) \leq \begin{cases} 1, & t < \mu_r^{1/r} \\ w_0, & t \geq \mu_r^{1/r}. \end{cases} \tag{3.10}$$

This inequality is sharp.

Proof. This theorem is a special case of Lemmas 3.4 and 3.4'. ||

Again we give a simple, direct proof different from that given for Lemma 3.4. Since F is IHR, $[1 - F(x)]^{1/x}$ is decreasing, and

$$\mu_r = \int_0^\infty rx^{r-1}[1 - F(x)]dx \geq \int_0^t rx^{r-1}[1 - F(t)]^{x/t}dx =$$

$$= rt^{r-1} \int_0^1 y^{r-1}[1 - F(t)]^y dy \equiv \varphi(1 - F(t)). \tag{3.11}$$

Differentiation easily yields the result that $\varphi(w)$ is strictly increasing in $[0,1]$; furthermore, $\varphi(0) = 0$, $\varphi(1) = t^r$. Since $t^r \geq \mu_r$, there exists a unique w_0 such that $\varphi(w_0) = \mu_r$. Monotonicity of φ together with $\varphi(1 - F(t)) \leq \mu_r$ implies that $w_0 \geq 1 - F(t)$. ||

Of course, bounds for distribution functions also yield bounds for percentiles. Specifically, for $0 < p < 1$, let ξ_p be a solution of

$$F(\xi_p -) \leq p \leq F(\xi_p)$$

(We assume F is right continuous). If $L(t) \leq F(t) \leq U(t)$, these

inequalities together imply $L(\xi_p -) \leq p \leq U(\xi_p)$, and if we define $L^{-1}(x) = \sup\{y : L(y) \leq x\}$, $U^{-1}(x) = \inf\{y : U(y) \geq x\}$, then $U^{-1}(p) \leq \xi_p \leq L^{-1}(p)$. Bounds for ξ_p obtainable in this way from (3.8) and (3.9) are given in

<u>Corollary 3.10</u>. If F is IHR, $F(0) = 0$ and $\int_0^\infty x^r dF(x) = \mu_r$, then

$$\mu_r^{1/r}\left[\int_0^1 ry^{r-1}(1-p)^y dy\right]^{-1/r} \geq \xi_p \geq \begin{cases} -\lambda_r^{1/r}\log(1-p), & p \leq 1 - \exp[\Gamma(r+1)]^{1/r} \\ \\ \mu_r^{1/r}, & p > 1 - \exp[\Gamma(r+1)]^{1/r} \end{cases} \tag{3.12}$$

where $\lambda_r = \mu_r/\Gamma(r+1)$.

<u>Proof</u>. The lower bound for ξ_p follows directly from (3.8) and the definition of U^{-1}. The upper bound follows from (3.11) with $t = \xi_p$, $1 - F(t) = p$. ||

Note that distributions which attain equality in (3.8) and (3.10) also attain equality in Corollary 3.10.

The case $p = 1/2$, $r = 1$ is of special interest, and yields

$$\mu_1 \log 2 \leq M \leq 2\mu_1 \log 2$$

where M is the median.

<u>Theorem 3.11</u>. If F is DHR, $F(0-) = 0$, $r > 0$ and $\int_0^\infty x^r dF(x) = \mu_r < \infty$, then

$$1 - F(t) \leq \begin{cases} e^{-t/\lambda_r^{1/r}}, & t \leq r\lambda_r^{1/r} \\ \\ \dfrac{r^r e^{-r}}{\Gamma(r+1)} \cdot \dfrac{\mu_r}{t^r} = r^r t^{-r} e^{-r}\lambda_r, & t \geq r\lambda_r^{1/r}. \end{cases} \tag{3.13}$$

This inequality is sharp.

Proof. We obtain the bound from Lemma 3.5, with $\zeta(x) = x^r$, and

$$\int_{0-}^{\infty} \zeta(x)dK_{\alpha;w}(x) = \alpha t^r \Gamma(r+1)(\log \alpha/w)^{-r},$$

so that

$$\psi(w) = \begin{cases} we^r t^r r^{-r}\Gamma(r+1), & w < e^{-r} \\ t^r\Gamma(r+1)(-\log w)^{-r}, & w \geq e^{-r}. \end{cases}$$

Computing $\psi^{-1}(\mu_r)$, we obtain (3.13) from (3.6). Sharpness follows from Lemma 3.5' .||

Theorem 3.12. Under the hypotheses of Theorem 3.11, no non-trivial lower bound for $1 - F(t)$ can be given.

Proof. This follows immediately from Lemma 3.7.||

3.2 BOUNDS FOR 1 - F, LAPLACE TRANSFORM GIVEN AT A POINT.

Bounds for $1 - F(t)$ under the assumption that $\int_{0-}^{\infty} e^{-sx}dF(x) = f^*(s)$ can be obtained even when the moments of F are not finite. Bounds of this kind do not seem to be generally known, although they are easily obtainable using standard methods.

We remark that inequalities given the first moment are obtained from those given $f^*(s)$ by letting $s \to 0$, $F(0-) = 0$.

Before giving the improved bounds for distributions with monotone hazard rate, we prove the following

Theorem 3.13. If $s > 0$, $\int_{0-}^{\infty} e^{-sx}dF(x) = f^*(s)$ and $s_0 = -s^{-1}\log f^*(s)$, then

$$1 - F(t-) \le \begin{cases} [1 - f^*(s)]/[1 - e^{-st}], & t \ge s_0 \\ 1, & t < s_0; \end{cases} \tag{3.14}$$

$$1 - F(t) \ge \begin{cases} 1 - [f^*(s)/e^{-st}], & t \le s_0 \\ 0, & t > s_0. \end{cases} \tag{3.15}$$

Proof. $1 - f^*(s) = \int_{0-}^{\infty}(1 - e^{-sx})dF(x) \ge \int_{t-}^{\infty}(1 - e^{-sx})dF(x) \ge (1 - e^{-st})\int_{t-}^{\infty} dF(x)$

$$= (1 - e^{-st})[1 - F(t-)]$$

which gives (3.14).

$$f^*(s) = \int_0^{\infty} e^{-sx}dF(x) \ge \int_0^{t+} e^{-sx}dF(x) \ge e^{-st}\int_0^{t+} dF(x) = e^{-st}F(t)$$

which gives (3.15). ||

Theorem 3.13'. Inequalities (3.14) and (3.15) are sharp.

Proof. For fixed s and t, we consider the following examples;

for (3.14), $t \ge s_0$, place probability p at t, $1 - p$ at 0 where

$$p = [1 - f^*(s)]/[1 - e^{-st}];$$

for (3.14), $t \le s_0$, place probability 1 at s_0;

for (3.15), $t \ge s_0$, place probability 1 at s_0;

for (3.15), $t \le s_0$, place probability p_m at t, $1 - p_m$ at m where

$$p_m = \frac{f^*(s) - e^{-sm}}{e^{-st} - e^{-sm}}. \text{ Then } \lim_{m\to\infty} p_m = \varphi(s)/e^{-st}.$$

In each case the distributions satisfy the hypothesis of Theorem 3.13. Equality is attained except in the last case, where the bound is approached asymptotically.

If F is IHR, it follows from (3.3) that

$f^*(s) = \int_0^\infty e^{-sx} dF(x) \leq (1 + \mu_1 s)^{-1}$ so that $f^*(s) < \infty$ for all $s > -\mu_1^{-1}$. Thus the following theorem has meaning for at least some values of $s < 0$.

Theorem 3.14. Let F be IHR, $F(0) = 0$, let $s \neq 0$, and let $f^*(s) = \int_0^\infty e^{-sx} dF(x) < \infty$.

Then

$$1 - F(t-) \geq \begin{cases} \exp\left[-\dfrac{st\, f^*(s)}{1 - f^*(s)}\right], & 0 < t \leq \dfrac{1 - f^*(s)}{s}, \\ \exp(L_0), & s^{-1}[1 - f^*(s)] < t \leq -s^{-1}\log f^*(s) \\ & \text{and } s > 0, \\ 0, & t > -s^{-1}\log f^*(s) \text{ and } s > 0; \text{ or} \\ & t > s^{-1}[1 - f^*(s)] \text{ and } s < 0, \end{cases} \tag{3.16}$$

where L_0 is the unique solution satisfying $-1 \leq L < 0$ of

$$f^*(s) = -L \exp(-st + 1 + L) .$$

The inequality is sharp.

Proof. Suppose first that $s > 0$. We note for later reference that

$$s^{-1}[1 - f^*(s)] \leq -s^{-1}\log f^*(s) \leq \mu_1 ;$$

the first inequality is the well-known inequality $\log x \leq x - 1$.

The second inequality follows from the fact that $f^*(r + s)$ is totally positive of order 2 in r and s (see Schoenberg (1951) for definition) so that $- s^{-1}\log f^*(s)$ is decreasing in s, and therefore $- s^{-1}\log f^*(s) \leq \lim_{s\to 0+} - s^{-1}\log f^*(s) = \mu_1$.

Now let $\zeta(x) = e^{-sx}$, so that the conditions of Lemma 3.3 are satisfied, and

$$f^*(s) \geq \inf_{0\leq z\leq t} \int_0^\infty e^{-sx}dG_z(x) = \inf_{0\leq z\leq t} \varphi(z)$$

where $\varphi(z) = -L\,e^{-sz}/[s(t - z) - L]$ and $L = \log[1 - F(t -)]$. Since e^{-sx} is convex, φ is also convex (See proof of Lemma 3.2). Setting $\frac{d}{dz}\varphi(z)\big|_{z=\tilde{z}} = 0$, we see that

$$\tilde{z} = t - (1 + L)/s.$$

Note that $\tilde{z} \leq t$ whenever $t \leq - s^{-1}\log f^*(s)$, since in this case $t \leq - s^{-1}\log f^*(s) \leq \mu_1$ implies $1 + L \geq 0$ by (3.8) with $r = 1$.

In case $s^{-1}[1 - f^*(s)] \geq t$, we claim $\tilde{z} \leq 0$. Suppose the contrary, $0 < \tilde{z} \leq t$. Then

$$f^*(s) \geq \inf_{0\leq z\leq t} \varphi(z) = \varphi(z_0) = - L \exp(- st + 1 + L), \qquad (3.17)$$

or

$$f^*(s)e^{st-1} \geq - L\, e^{L},$$

and since $z_0 > 0$, $- L > 1 - st$. But $1 + L \geq 0$ so that $1 - st < - L \leq 1$. Hence $f^*(s)e^{st-1} \geq - L e^{L} > (1 - st)e^{st-1}$,

or $f^*(s) > 1 - st$, contradicting $t \leq s^{-1}[1 - f^*(s)]$. Thus $z \leq 0$ and we conclude $\inf\limits_{0 \leq z \leq t} \varphi(z) = \varphi(0)$. But $f^*(s) \geq \varphi(0)$ yields (3.16) for $t \leq s^{-1}[1 - f^*(s)]$ and $s > 0$.

Next, suppose that $s^{-1}[1 - f^*(s)] < t \leq - s^{-1}\log f^*(s)$. The function $xe^{-x+st-1}$ is monotone increasing (decreasing) in $x \epsilon [0,1]$(in $[1,\infty)$), and attains the maximum e^{st-1} at $x = 1$. Since $t \leq - s^{-1}\log f^*(s)$, there exist solutions $0 < c_0 < 1$, $c_1 > 1$ of $f^*(s) = xe^{-x+1-st}$, and setting $c = -L$ we obtain from $f^*(s) \geq ce^{-c+1-st}$ (i.e.,(3.17)) that $c \leq c_0$ or $c \geq c_1$. But $1 + L > 0$ implies $c < 1$, so that $c \leq c_0$. This yields (3.16) in case $s^{-1}[1 - f^*(s)] < t \leq - s^{-1}\log f^*(s)$ and $s > 0$.

If $s < 0$, then let $\zeta(x) = e^{-sx} - 1$, and the inequality follows from Lemmas 3.1 and 3.2.

Sharpness of (3.16) follows from Lemma 3.1', and its analog giving sharpness of Lemma 3.3.||

<u>Theorem 3.15</u>. If F is IHR, $F(0) = 0$, $s \neq 0$ and $\int_0^\infty e^{-sx}dF(x) = f^*(s) < \infty$, then

$$1 - F(t) \leq \begin{cases} 1, & t < - s^{-1}\log f^*(s) \\ e^{-a_0 t}, & t \geq - s^{-1}\log f^*(s) \end{cases} \qquad (3.18)$$

where a_0 is the unique solution of

$$f^*(s) = \frac{s}{s+a} e^{-(s+a)t} + \frac{a}{s+a} .$$

The inequality is sharp.

Proof. In case $s < 0$, the inequality follows from Lemma 3.4, and for $s > 0$, it follows from the analog of Lemma 3.4 for decreasing ζ. Sharpness follows from Lemma 3.4'.||

Theorem 3.16. If F is DHR, $F(0-) = 0$, $s \neq 0$ and $\int_0^\infty e^{-sx}dF(x) = f^*(s) < \infty$, then

$$1 - F(t) \leq \begin{cases} \exp\{-st\, f^*(s)/[1 - f^*(s)]\}, & t \leq [1 - f^*(s)]/s \\ e^{st-1}[1 - f^*(s)]/st, & [1 - f^*(s)]/s \leq t \text{ and } st < 1 \\ 1 - f^*(s), & st \geq 1 . \end{cases} \tag{3.19}$$

The inequality is sharp.

Proof. We compute $\int_{0-}^\infty e^{-sx}dK_\alpha(x) = \alpha\eta/(st + \eta) + 1 - \alpha$ where $\eta = \log \alpha - \log[1 - F(t)]$. The inequality then follows from Lemma 3.5 and its analog for decreasing ζ. Sharpness follows from Lemma 3.5'. In case $t \leq [1 - f^*(s)]/s$, the inequality is more easily obtained as follows. From the proof of Lemma 3.5 and its analog, $sf^*(s) \leq s \int_0^\infty e^{-sx}dK_\alpha(x)$ for some α, $1 - F(t) < \alpha \leq 1$; solving this for $1 - F(t)$ yields $1 - F(t) \leq \sup_{1-F(t)<\alpha\leq 1} \alpha \exp\{[st(f^* - 1) + \alpha\, st]/[f^* - 1]\}$. From this one easily obtains (3.19) for $st < 1$. ||

If $st \geq 1$, the distribution achieving equality in (3.19) is improper, but can be approximated by proper distribution functions.

Theorem 3.17. If F is DHR, $F(0-) = 0$, $s > 0$ and $\int_0^\infty e^{-sx}dF(x) = f^*(s) < \infty$, then

$$1 - F(t) \geq e^{L_0} \tag{3.20}$$

where L_0 is the unique solution of

$$f^*(s) = L(1 - e^{L-st})/(L - st).$$

The inequality is sharp.

Proof. This is a direct consequence of Lemmas 3.6 and 3.6'.||

Note that by Lemma 3.7, a non-trivial lower bound cannot be given under the conditions of Theorem 3.17 if $s < 0$.

3.3 BOUNDS FOR 1 - F UTILIZING BOUNDS ON THE HAZARD RATE.

In this section we indicate, without any attempt at generality, how bounds on the hazard rate can be used to yield bounds for the distribution function. Theorems 3.19 and 3.21 which assume an increasing hazard rate have unstated DHR analogs. We assume that F has a density f, so that the hazard rate $q(x) = f(x)/[1 - F(x)]$ is defined.

Theorem 3.18. If $F(0-) = 0$, $q(x) \geq \alpha$ for all $x \geq 0$, and $\int_x^\infty xf(x)dx = \mu$, then

$$1 - F(t) \leq \begin{cases} e^{-\alpha t}, & t \leq -\frac{1}{\alpha}\log(1 - \alpha\mu) = t_0 \\ \dfrac{\alpha\mu e^{-\alpha t}}{1 - e^{-\alpha t}} & t \geq t_0 \,; \end{cases} \tag{3.21}$$

$$1 - F(t) \geq \begin{cases} \alpha\mu - 1 + e^{-\alpha t}, & t \leq t_0 \\ 0, & t \geq t_0 \,. \end{cases} \tag{3.22}$$

We remark that $q(x) \geq \alpha$ implies $\alpha\mu \leq 1$ so that t_0 is defined. More generally, by integrating both sides of $x^r f(x) \geq \alpha x^r [1 - F(x)]$ it follows that

$$\mu_r \geq \alpha\mu_{r+1}/(r + 1), \qquad r > -1 . \tag{3.23}$$

It will be seen from the proof that the bound $1 - F(t) \leq e^{-\alpha t}$ is valid for all t; this is a sharp bound for all t in case μ is unknown.

<u>Proof of (3.21)</u>. $q(x) \geq \alpha$ implies $\int_0^t q(x)dx \geq \alpha t$ implies $1 - F(t) = \exp(-\int_0^t q(x)dx) \leq e^{-\alpha t}$, which is the upper bound for $t \leq t_0$. To obtain the upper bound for $t \geq t_0$, note first that $q(w) \geq \alpha$ implies for $t > x$,

$$[1 - F(t)]/[1 - F(x)] = \exp(-\int_x^t q(w)dw) \leq e^{-\alpha(t-x)}.$$

Thus

$$\mu = \int_0^\infty xf(x)dx = \int_0^t xf(x)dx + \int_t^\infty xf(x)dx \geq \int_0^t x\alpha[1 - F(x)]dx + t[1 - F(t)] \geq$$

$$\geq \int_0^t x\alpha[1 - F(t)]e^{\alpha(t-x)}dx + t[1 - F(t)] =$$

$$= \alpha[1 - F(t)]e^{\alpha t}[-t\alpha^{-1}e^{-\alpha t} + \alpha^{-2}(1 - e^{-\alpha t})] + t[1 - F(t)] =$$

$$= [1 - F(t)][1 - e^{-\alpha t}]/\alpha e^{-\alpha t} ,$$

or

$$1 - F(t) \leq \alpha\mu e^{-\alpha t}/(1 - e^{-\alpha t}). \;\|$$

Proof of (3.22).

$$\mu = \int_0^\infty [1 - F(x)]dx = \int_0^t [1 - F(x)]dx + \int_t^\infty [1 - F(x)]dx$$

$$\leq \int_0^t e^{-\alpha x}dx + \int_t^\infty \frac{f(x)}{\alpha}\, dx = \frac{1 - e^{-\alpha t}}{\alpha} + \frac{1 - F(t)}{\alpha} ,$$

or

$$1 - F(t) \geq \alpha\mu - 1 + e^{-\alpha t} . \quad \|$$

Theorem 3.18'. Equality is attained in (3.21) uniquely by the distribution

$$1 - G(x) = \begin{cases} e^{-\alpha x}, & 0 \leq x \leq t_0 , \\ 0, & x > t_0 , \end{cases} \qquad t \leq t_0 ;$$

$$1 - G(x) = \begin{cases} \dfrac{\alpha\mu e^{-\alpha x}}{1 - e^{-\alpha t}} , & x \leq t , \\ 0 , & x > t , \end{cases} \qquad t \geq t_0 .$$

Equality is attained in (3.22), uniquely when $t < t_0$, by the distribution

$$1 - G(x) = \begin{cases} e^{-\alpha x} , & x < t \\ (\alpha\mu - 1 + e^{-\alpha t})e^{-\alpha(x-t)}, & x \geq t , \end{cases} \qquad t \leq t_0 ;$$

$$1 - G(x) = \begin{cases} e^{-\alpha x}, & 0 \leq x \leq t_0 , \\ 0 , & x > t_0 , \end{cases} \qquad t \geq t_0 .$$

The remark following Lemma 3.4' is appropriate to the above left continuous distributions. The above distributions do not have densities, but can be approximated by distributions satisfying the hypotheses of Theorem 3.18 and having densities. Inequalities (3.21) and (3.22) hold when no density exists, providing $\lim\limits_{\Delta \to 0} [F(x+\Delta) - F(x)]/\Delta[1 - F(x)] \geq \alpha$.

Note that if $t \leq t_0$, under the hypotheses of Theorem 3.18,

$$f(t) \geq \alpha[1 - F(t)] \geq \alpha(\alpha\mu - 1 + e^{-\alpha t}). \tag{3.24}$$

Theorem 3.19 If $F(0-) = 0$, $q(x) \geq \alpha$, $q(x)$ is increasing and $\int_0^\infty xf(x)dx = \mu$, then

$$1 - F(t) \leq \begin{cases} e^{-\alpha t}, & t \leq -\frac{1}{\alpha}\log(1 - \alpha\mu) = t_0 , \\ e^{-yt} , & t \geq t_0 , \end{cases} \tag{3.25}$$

where y is determined by $(1 - e^{-yt})/y = \mu$;

$$1 - F(t) \geq \begin{cases} e^{-t/\mu} , & t \leq \mu , \\ e^{-(\alpha z+1)}, & \mu < t < t_0 , \\ 0 , & \mu \geq t_0 , \end{cases} \tag{3.26}$$

where z is determined by $1 - \alpha\mu = [1 - \alpha(t - z)]e^{-\alpha z}$.

Proof of (3.25). For $t \leq t_0$, (3.25) follows from (3.21); for $t \geq t_0$, (3.25) follows from (3.10) with $r = 1$. ||

Proof of (3.26). For $t < \mu$, (3.26) follows from (3.8) with $r = 1$. To obtain the bound for $\mu < t < t_0$, note first that $q(x) \geq \alpha$ implies $\log[1 - F(x)] \leq -\alpha x$. Since $\log[1 - F(x)]$ is concave, there exists z, $0 \leq z \leq t$, such that

$$\log[1 - F(x)] \leq \begin{cases} -\alpha x, & 0 \leq x \leq z \\ \\ \dfrac{\alpha z - (\alpha t + A)}{t - z}(x - z) - \alpha z, & x \geq z, \end{cases}$$

where $\log[1 - F(t)] = -(\alpha t + A)$.

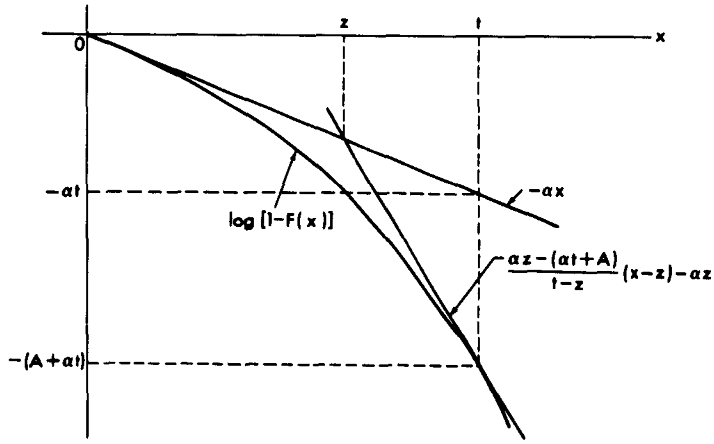

Figure 3.1

Thus for some z, $0 \leq z \leq t$,

$$\mu = \int_0^\infty [1 - F(x)]dx \leq \int_0^z \exp(-\alpha x)dx + \int_z^\infty \exp[\frac{\alpha z - (\alpha t + A)}{t - z}(x - z) - \alpha z]dx =$$

$$= \frac{1 - e^{-\alpha z}}{\alpha} - \frac{t - z}{\alpha z - (\alpha t + A)} e^{-\alpha z},$$

or

$$\frac{\mu - \alpha^{-1}(1 - e^{-\alpha z})}{(t - z)e^{-\alpha z}} \leq \frac{1}{A + \alpha(t - z)} .$$

Since $z \leq t < t_0$, $\mu - \alpha^{-1}(1 - e^{-\alpha z}) > 0$ and

$$A \leq \frac{(t - z)e^{-\alpha z}}{\mu - \alpha^{-1}(1 - e^{-\alpha z})} - \alpha(t - z) \equiv \varphi(z) .$$

$$\varphi'(z) = \alpha - \frac{[\mu - \alpha^{-1}(1 - e^{-\alpha z})][\alpha(t - z)e^{-\alpha z} + e^{-\alpha z}] - (t - z)e^{-2\alpha z}}{[\mu - \alpha^{-1}(1 - e^{-\alpha z})]^2} ,$$

and $\varphi'(z) = 0$ if and only if $1 - a\mu = [1 - \alpha(t - z)]e^{-\alpha z} \equiv \psi(z)$. $\psi(z)$ is increasing in z; $\psi(0) = 1 - \alpha t \leq 1 - \alpha\mu$ since $t \geq \mu$; $\psi(t) = e^{-\alpha t} > 1 - \alpha\mu$ since $t < t_0$. Thus for some z_0, $0 \leq z_0 \leq t$, $\varphi'(z_0) = 0$. Since $\varphi(z_0) \geq \varphi(z)$, $0 \leq z \leq t$, $A \leq \varphi(z_0)$, or

$$A + \alpha t \leq \alpha z_0 + \frac{(t - z_0)e^{-\alpha z_0}}{\mu - \alpha^{-1}(1 - e^{-\alpha z_0})} = \alpha z_0 + 1. \quad \|$$

<u>Theorem 3.19'</u>. Equality is attained in (3.25) uniquely by the distribution

$$1 - G(x) = \begin{cases} e^{-\alpha x}, & 0 \leq x \leq t_0 , \\ 0, & x > t_0 , \end{cases} \qquad t \leq t_0 ;$$

$$1 - G(x) = \begin{cases} e^{-xy}, & x \leq t , \\ 0 , & x > t , \end{cases} \qquad t > t_0 \quad .$$

Equality is attained in (3.26), uniquely when $t < t_0$, by the distribution

$$1 - G(x) = e^{-x/\mu}, \quad x \geq 0 , \qquad t \leq \mu ;$$

$$1 - G(x) = \begin{cases} e^{-\alpha x}, & 0 \leq x \leq z_0, \\ \exp\left(- \dfrac{x - z_0}{t - z_0} - \alpha z_0\right), & x > z_0, \end{cases} \qquad \mu < t < t_0 ;$$

$$1 - G(x) = \begin{cases} e^{-\alpha x} , & 0 < x \leq t_0 , \\ 0 , & x > t_0 , \end{cases} \qquad t > t_0 \quad .$$

Proof. For equality in (3.25) and $t > t_0$, G has hazard rate

$$q_G(x) = \begin{cases} y , & x \leq t \\ \infty , & x > t \end{cases} \quad .$$

To see that $y \geq \alpha$, let $\theta(y,t) = (1 - e^{-yt})/y$. Then $\partial\theta/\partial y \leq 0$, $\partial\theta/\partial t \geq 0$, and $t = t_0$ implies $y = \alpha$. Therefore if $\theta(y,t) = \mu$ and $t \geq t_0$, $y \geq \alpha$.

For equality in (3.26) and $\mu < t < t_0$, G has hazard rate

$$q_G(x) = \begin{cases} \alpha, & x \leq z, \\ (t-z)^{-1}, & x > z. \end{cases}$$

Since $\alpha\mu < 1$, $(1-\alpha\mu)e^{\alpha z} = 1 - \alpha(t-z) > 0$, or $(t-z)^{-1} \geq \alpha$. ||

Theorem 3.20. If $F(0-) = 0$, $q(x) \leq \beta < \infty$ for all $x \geq 0$, and $\int_0^\infty xf(x)dx = \mu$, then

$$1 - F(t) \leq \begin{cases} e^{-\beta z_0}, & t > \mu - \beta^{-1}, \\ 1, & t \leq \mu - \beta^{-1}, \end{cases} \tag{3.27}$$

where z_0 is the unique solution of $(t-z)e^{-\beta z} = \mu - \beta^{-1}$ satisfying $0 \leq z_0 \leq t$;

$$1 - F(t) \geq e^{-\beta t}. \tag{3.28}$$

Proof of (3.28). $q(x) \leq \beta$ implies $\int_0^t q(x)dx \leq \beta t$ implies $1 - F(t) = \exp(-\int_0^t q(x)dx) \geq e^{-\beta t}$. ||

Proof of (3.27). From (3.28), it follows that if $z > 0$,

$$\int_0^z [1 - F(x)]dx \geq \int_0^z e^{-\beta x}dx = (1 - e^{-\beta z})/\beta .$$

Since $q(x) \leq \beta$, $1 - F(x) \geq \beta^{-1}f(x)$, and $\int_t^\infty [1 - F(x)]dx \geq \beta^{-1}[1 - F(t)]$.

Thus for $0 < z < t$,

$$\mu = \int_0^z [1 - F(x)]dx + \int_z^t [1 - F(x)]dx + \int_t^\infty [1 - F(x)]dx \geq$$

$$\geq \beta^{-1}(1 - e^{-\beta z}) + (t - z)[1 - F(t)] + \beta^{-1}[1 - F(t)], \quad \text{or}$$

$1 - F(t) \leq [\beta\mu - 1 + e^{-\beta z}]/[\beta(t - z) + 1] \equiv \varphi(z)$. Setting $\varphi'(z) = 0$ in order to minimize the bound, we obtain

$$\mu - \beta^{-1} = (t - z)e^{-\beta z} \equiv \psi(z).$$

From the facts that $\psi(z)$ is decreasing in $0 \leq z \leq t$, $\psi(t) = 0$, $\psi(0) = t$, it follows that for $t \geq \mu - \beta^{-1}$ the equation $\psi(z) = \mu - \beta^{-1}$ has a unique solution z_0 satisfying $0 \leq z_0 \leq t$. To complete the proof, note that since $(t - z_0)e^{-\beta z_0} = \mu - \beta^{-1}$,
$\varphi(z_0) = (\beta\mu - 1 + e^{-\beta z_0}) / [(\beta\mu - 1)e^{\beta z_0} + 1] = e^{-\beta z_0}$. ||

<u>Theorem 3.20'</u>. Equality is attained in (3.27), uniquely when $t > \mu - \beta^{-1}$, by the distribution

$$1 - G(x) = \begin{cases} e^{-\beta x}, & 0 \leq x \leq z_0, \\ e^{-\beta z_0}, & z_0 < x < t, \\ e^{-\beta(x-t+z_0)}, & x \geq t, \end{cases} \qquad t > \mu - \beta^{-1}$$

$$1 - G(x) = \begin{cases} 1, & x \leq t \\ e^{-a(x-t)}, & x > t \end{cases}$$

Thus for $0 < z < t$,

$$\mu = \int_0^z [1 - F(x)]dx + \int_z^t [1 - F(x)]dx + \int_t^\infty [1 - F(x)]dx \geq$$

$$\geq \beta^{-1}(1 - e^{-\beta z}) + (t - z)[1 - F(t)] + \beta^{-1}[1 - F(t)], \quad \text{or}$$

$1 - F(t) \leq [\beta\mu - 1 + e^{-\beta z}]/[\beta(t - z) + 1] \equiv \varphi(z)$. Setting $\varphi'(z) = 0$ in order to minimize the bound, we obtain

$$\mu - \beta^{-1} = (t - z)e^{-\beta z} \equiv \psi(z).$$

From the facts that $\psi(z)$ is decreasing in $0 \leq z \leq t$, $\psi(t) = 0$, $\psi(0) = t$, it follows that for $t \geq \mu - \beta^{-1}$ the equation $\psi(z) = \mu - \beta^{-1}$ has a unique solution z_0 satisfying $0 \leq z_0 \leq t$. To complete the proof, note that since $(t - z_0)e^{-\beta z_0} = \mu - \beta^{-1}$, $\varphi(z_0) = (\beta\mu - 1 + e^{-\beta z_0}) / [(\beta\mu - 1)e^{\beta z_0} + 1] = e^{-\beta z_0}$. ||

Theorem 3.20'. Equality is attained in (3.27), uniquely when $t > \mu - \beta^{-1}$, by the distribution

$$1 - G(x) = \begin{cases} e^{-\beta x}, & 0 \leq x \leq z_0 , \\ e^{-\beta z_0}, & z_0 < x < t , \\ e^{-\beta(x-t+z_0)}, & x \geq t, \end{cases} \qquad t > \mu - \beta^{-1}$$

$$1 - G(x) = \begin{cases} 1 , & x \leq t \\ e^{-a(x-t)}, & x > t \end{cases}$$

where $a^{-1} = \mu - t > \beta^{-1}$. Equality is attained in (3.28) by the distribution

$$1 - G(x) = \begin{cases} e^{-\beta x}, & 0 < x \leq t \\ e^{-t(\beta-a)-ax}, & x > t\ , \end{cases}$$

where $a = e^{-\beta t}/[\mu - \beta^{-1}(1 - e^{-\beta t})]$.

We omit a proof of this theorem.

If $t > \mu - \beta^{-1}$, then (3.23) yields

$$f(t) \leq \beta[1 - F(t)] \leq \beta e^{-\beta z_0}\ . \tag{3.29}$$

In place of (3.27), it is possible to give an explicit upper bound for $1 - F(t)$;

$$1 - F(t) \leq \mu/(t + \beta^{-1}). \tag{3.30}$$

To obtain (3.30), note that

$$\mu = \int_0^t [1 - F(x)]dx + \int_t^\infty [1 - F(x)]dx \geq t[1 - F(t)] + \int_t^\infty \beta^{-1} f(x)dx =$$

$$= (t + \beta^{-1})[1 - F(t)].$$

This improvement of Markov's inequality is of course not sharp.

The hypotheses of Theorem 3.20 yields the moment inequality.

$$\mu_r \leq \beta\mu_{r+1} /(r + 1), \qquad r > -1 \tag{3.31}$$

which is to be compared with (3.23).

Theorem 3.21. If $F(0-) = 0$, $q(x) \leq \beta$, q is increasing and $\int_0^\infty xf(x)dx = \mu$, then

$$1 - F(t) \leq \begin{cases} 1, & t \leq \mu - \beta^{-1}, \\ w_0, & t > \mu - \beta^{-1}. \end{cases} \tag{3.32}$$

where w_0 is the unique solution of $\mu = -[t(1 - w)/\log w] + w/\beta$;

$$1 - F(t) \geq \begin{cases} e^{-t/\mu}, & t \leq \mu, \\ e^{-\beta(t-\mu)-1}, & t > \mu. \end{cases} \tag{3.33}$$

Proof of (3.32). If $L = \log[1 - F(t)]$ then since $\log[1 - F(x)]$ is concave, $1 - F(x) \geq e^{Lx/t}$, $x \leq t$. Since $q(x) \leq \beta$, $1 - F(x) \geq f(x)/\beta$, so that

$$\mu = \int_0^\infty [1 - F(x)]dx \geq \int_0^t \exp(Lx/t)dx + \int_t^\infty \beta^{-1} (x)dx = L^{-1}t(e^L - 1) + \beta^{-1}[1 - F(t)]$$

$$= - \frac{tF(t)}{\log[1 - F(t)]} + \frac{1 - F(t)}{\beta} \equiv \varphi(1 - F(t)).$$

Since $\lim_{w \to 0} \varphi(w) = 0$, $\lim_{w \to 1} \varphi(w) = \beta^{-1} + t$ and $\varphi(w)$ is increasing in w, there exists a unique w_0 satisfying $\varphi(w_0) = \mu$ whenever $t > \mu - \beta^{-1}$. Furthermore, $1 - F(t) \leq w_0$. ||

Proof of (3.33) . Again let $L = \log[1 - F(t)]$. Since $q(x)$ is increasing, there exists z such that $\log[1 - F(x)] \leq L \frac{x - z}{t - z}$, $x \geq z$. Since $q(x) \leq \beta$, and $F(0 -) = 0$, it follows that $0 \leq z \leq t + L\beta^{-1}$. Thus for some z, $0 \leq z < t + L\beta^{-1}$,

$$1 - F(x) \leq \begin{cases} 1 , & x < z \\ \exp(L\frac{x - z}{t - z}), & x \geq z, \end{cases}$$

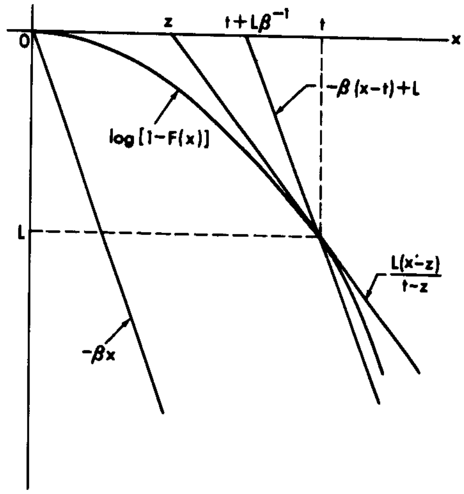

Figure 3.2

and

$$\mu = \int_0^\infty [1 - F(x)]dx \leq z + \int_z^\infty \exp(L \frac{x - z}{t - z})dz = z - (t - z)L^{-1} .$$

Since $t > z$, $\psi(z) = (\mu - z)/(t - z) \leq - L^{-1}$. $\psi'(z) = (\mu - t)/(t - z)^2$, so that if $t \geq \mu$ $\psi(z)$ is decreasing and $\min_{0 \leq z \leq t+L\beta^{-1}} \psi(z) =$ $= \psi(t + L\beta^{-1}) \leq - L^{-1}$, or $L \geq \beta(\mu - t) - 1$. In case $t \leq \mu$ the bound follows from (3.8) with $r = 1$.||

<u>Theorem 3.21'</u>. Equality is attained in (3.32), uniquely for $t > \mu - \beta^{-1}$, by the distribution

$$1 - G(x) = \begin{cases} 1\ , & 0 \le x \le t \\ \exp(-\frac{x - t}{\mu - t}), & x > t\ , \end{cases} \qquad t \le \mu - \beta^{-1}$$

$$1 - G(x) = \begin{cases} w_0^{x/t}, & 0 \le x \le t \\ w_0 e^{-\beta(x-t)}, & x > t, \end{cases} \qquad t > \mu - \beta^{-1}.$$

Equality is attained in (3.33) uniquely by the distribution

$$1 - G(x) = e^{-x/\mu}, \qquad t \le \mu$$

$$1 - G(x) = \begin{cases} 1, & x \le \mu - \beta^{-1} \\ \exp[-\beta(x - \mu) - 1], & x \ge \mu - \beta^{-1}, \end{cases} \qquad t > \mu.$$

In the case of (3.32), $t > \mu - \beta^{-1}$, G has hazard rate

$$q_G(x) = \begin{cases} -t^{-1}\log w_0\ , & 0 \le x \le t \\ \beta, & x > t\ , \end{cases}$$

and we note that $-t^{-1} \log w_0 = (1 - w_0)(\mu - w_0 / \beta)^{-1} \le \beta$ since $\beta\mu \ge 1 \ge w_0$.

3.4 BOUNDS IN TERMS OF PERCENTILES.

From the general results of Section 3, bounds for expectations of monotone functions can be obtained in terms of percentiles. In particular, it follows from (3.2) that if F is IHR with $F(0) = 0$ and if ζ is a function increasing on $[0,\infty)$, then

$$\int_0^\infty \zeta(x)dF(x) \leq \sup_{0\leq z\leq t} \int_0^\infty \zeta(x)dG_z(x) \tag{3.34}$$

where

$$G_z(x) = \begin{cases} 1, & x \leq z \\ [1 - F(t-)]^{(x-z)/(t-z)}, & x \geq z. \end{cases}$$

Since G_z has a density that is a Pólya frequency function of order 2 (PF_2) (see Section 5 for a definition), (3.34) is also sharp with this strengthened hypothesis.

With $\zeta(x) = \chi_{[s,\infty)}(x)$, the characteristic function of $[s,\infty)$, it follows from (3.34) that

$$1 - F(s) \leq \begin{cases} [1 - F(t)]^{s/t}, & s \geq t \\ 1, & s < t; \end{cases} \tag{3.35}$$

this bound is also given by Barlow Marshall and Proschan (1963). Here the exponential and degenerate distributions achieve equality.

By interchanging s and t in (3.35) it follows that

$$1 - F(s) \geq \begin{cases} [1 - F(t)]^{s/t}, & s \leq t \\ 0, & s > t \end{cases} \tag{3.36}$$

More generally, let

$$1 - H(x) \begin{cases} [1 - F(t)]^{x/t}, & x \leq t \\ 0, & x > t. \end{cases}$$

Then by (3.35), $1 - H(x) \leq 1 - F(x)$, $x \leq t$, so that if ζ is increasing,

$$\int_0^\infty \zeta(x)dF(x) \geq \int_0^{t+} \zeta(x)dF(x) \geq \int_0^{t+} \zeta(x)dH(x) = \int_0^\infty \zeta(x)dH(x). \tag{3.37}$$

With $\zeta(x) = \chi_{[s,\infty)}$, (3.37) reduces to (3.36).

Note that equality in (3.37) is attained by the distribution function H which does not have a PF_2 density, so that (3.37) can be improved in case F has a PF_2 density. Such an improvement is given by (5.5).

4. BOUNDS FOR $1 - F$ GIVEN TWO MOMENTS.

In this section we confine our attention to "power moments", although the methods used are more generally applicable.

In order to illustrate these methods, we give a heuristic discussion of the following problem: obtain sharp upper and lower bounds for $1 - F(t)$ when $F \epsilon \mathcal{F}$, the class of IHR distributions satisfying $F(0) = 0$, $F(m) = 1$ and $\int_0^m x dF(x) = \mu_1$ ($= 1$ for convenience). Let $\mathcal{G}_1 = \{G_w: 0 \leq w \leq 1\}$, where

$$1 - G_w(x) = \begin{cases} 1, & x \leq w \\ e^{-a(x-w)}, & w \leq x < m \\ 0 \quad , & x \geq m \end{cases} \tag{4.1}$$

and a is determined by

$$a^{-1}(1 - e^{-a(m-w)}) = 1 - w .$$

Let $\mathcal{G}_2 = \{G_w: 1 \leq w \leq m\}$, where

$$1 - G_w(x) = \begin{cases} e^{-bx} , & 0 \leq x < - b^{-1} \log(1 - b) = w \\ \\ 0, & x \geq w. \end{cases} \tag{4.2}$$

Note that $\mathcal{G}_i \subset \mathcal{F}$. If we show that

$$\{(x, 1 - G(x)): G \epsilon \mathcal{G}_1 \cup \mathcal{G}_2 \} = \{(x, 1 - F(x)): F \epsilon \mathcal{F}\},$$

then for $F \in \mathcal{F}$ and fixed $t \in [0,m]$,

$$\inf\ 1 - G(t) \leq 1 - F(t) \leq \sup\ 1 - G(t), \qquad (4.3)$$

where the extremums are taken over $\mathcal{B}_1 \cup \mathcal{B}_2$.

Let the $F \in \mathcal{F} - (\mathcal{B}_1 \cup \mathcal{B}_2)$. Since F and G_w have the same mean, they cross at least once; since F is IHR, they cross at most twice in $(0,m)$. If there are two such crossings, say at u and $v > u$, then $1 - F(x) > 1 - G_w(x)$ for $u < x < v$. If $w \geq 1$, there is exactly one crossing for $x < w$, say at v, and $1 - F(x) > 1 - G_w(x)$, $0 < x < v$. We remark that since F is IHR, it is continuous except possibly for a jump at the right-hand endpoint of its support.

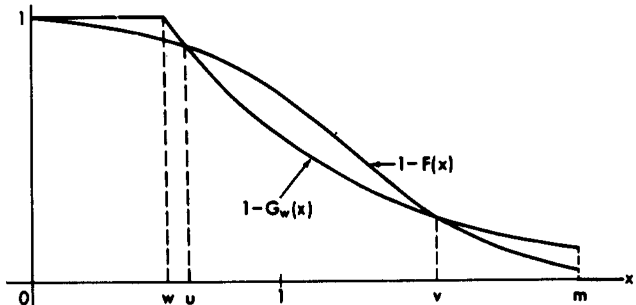

Figure 4.1

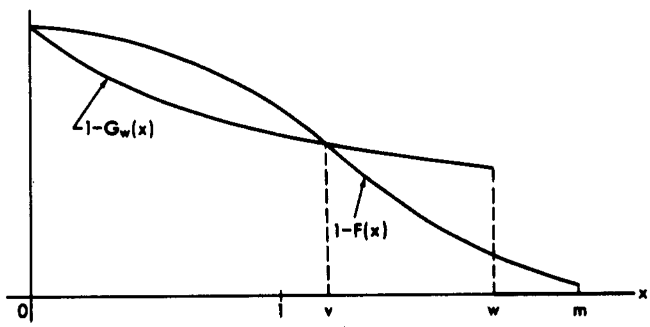

Figure 4.2

Let v_w be the crossing from above of $1 - G_w(x)$ by $1 - F(x)$ if it exists; otherwise let $v_w = m$. Then $v_m \leq m$, and as $w = - b^{-1} \log(1 - b)$ decreases from m to 1, b decreases to 0, and heurstically, v_m decreases to 0. Again consider $v_m = v_0(G_m = G_0)$ and increase w from 0 to 1. Then v_w increases to m as w increases to 1. This shows that for fixed $F \epsilon \mathcal{F}$, $(x, 1 - F(x)) \epsilon \{(x, 1 - G_w(x)), 0 \leq w \leq m\}$ for all x, $0 \leq x \leq m$.

The bound $\sup_{G \epsilon \mathcal{G}_1 \cup \mathcal{G}_2} [1 - G(t)]$ is a special case of (3.9). In case $m = \infty$, $a = (1 - w)^{-1}$ and if $t < 1$ $\inf_{G \epsilon \mathcal{G}_1 \cup \mathcal{G}_2} [1 - G(t)] = = \min_{0 \leq w \leq 1} \exp(- \frac{t - w}{1 - w}) = e^{-t}$, which is (3.8) with $r = 1$.

Note that for G_w in $\mathcal{G}_1$ and $m = \infty$,

$$\mu_2 = 2 \int_0^\infty x[1 - G_w(x)]dx = 1 + (1 - w)^2,$$

and as w ranges over $[0,1]$, μ_2 ranges over $[1,2]$. But for any IHR distribution, $\mu_1^2 \leq \mu_2 \leq 2\mu_1^2$, and since $1 - G_w(1) = e^{-1}$ for all w, we see that e^{-1} is a sharp lower bound for $1 - F(\mu_1 -)$ whenever F is IHR with mean $\mu_1 = 1$, regardless of the specified value of μ_2.

4.1 BOUNDS FOR $1 - F$ WHEN F IS IHR

Retain the assumption that $\mu_1 = 1$, fix μ_2, and let $T_0 = 1 - (\mu_2 - 1)^{1/2}$, $T_1 = - a_0^{-1} \log(1 - a_0)$ where a_0 in $[0,1]$ satisfies

$$\varphi(a) \equiv \frac{2}{a} [1 + \frac{1 - a}{a} \log(1 - a)] = \mu_2 \quad .$$

Such a solution exists since $\varphi(a)$ is continuous in a, $\lim_{a \to 0} \varphi(a) = 1$, and $\lim_{a \to 1} \varphi(a) = 2$ (F IHR implies $1 \leq \mu_2 \leq 2$).

Let $\mathcal{B}_3 = \{G_T: T \geq T_1\}$, where

$$1 - G_T(x) = \begin{cases} 1, & x < \Delta \\ e^{-a(x-\Delta)}, & \Delta \leq x < T \\ 0, & x > t \end{cases} , \qquad T \geq T_1 \quad , \tag{4.4}$$

$$1 = \Delta + \frac{1}{a} [1 - e^{-a(t-\Delta)}] \; , \tag{4.5}$$

and

$$\frac{\mu_2}{2} = \frac{\Delta^2}{2} - \frac{aT + 1}{a^2} e^{-a(T-\Delta)} + \frac{a\Delta + 1}{a^2} \; . \tag{4.6}$$

Let $\mathcal{B}_4 = \{G_T: T_0 \leq T \leq T_1\}$, where

$$1 - G_T(x) = \begin{cases} e^{-a_1 x}, & x \leq T \\ e^{-a_1 T - a_2(x-T)}, & x \geq T \end{cases} , \quad T_0 \leq T \leq T_1 \; , \tag{4.7}$$

$$1 = a_1^{-1}(1 - e^{-a_1T}) + a_2^{-1}\, e^{-a_1T}\,, \tag{4.8}$$

and

$$\frac{\mu_2}{2} = a_1^{-2}\left[1 - (1 + a_1T)e^{-a_1T}\right] + a_2^{-2}(1 + a_2T)e^{-a_1T}. \tag{4.9}$$

We defer the proof that solutions of (4.5),(4.6), and (4.8) exist .

A principal result of this section is

Theorem 4.1. Let $\mathcal{F}_2$ be the class of IHR distributions F such that $F(0) = 0$, $\int_0^\infty x\,dF(x) = 1$ and $\int_0^\infty x^2 dF(x) = \mu_2$. Then

$$\{(x, 1 - F(x)) : F \in \mathcal{F}_2\} = \{(x, 1 - G(x)) : G \in \mathcal{G}_3 \cup \mathcal{G}_4\}\,,$$

and hence

$$\inf[1 - G(t)] \le 1 - F(t) \le \sup\,[1 - G(t)], \tag{4.10}$$

where the extremums are taken over $\mathcal{G}_3 \cup \mathcal{G}_4$.

Equations (4.5),(4.6), and (4.7),(4.8) guarantee that G_T has mean $\mu_1 = 1$, and second moment μ_2 so that $\mathcal{G}_3 \cup \mathcal{G}_4 \subset {}_2$. Hence it is clear that (4.10) is sharp, although it is not clear that equality is attainable.

We defer the proof of Theorem 4.1.

Remark. We use repeatedly the fact that the functions $c_1e^{-b_1x}$, $c_2e^{-b_2x}$ are identical or have at most a single crossing (simple intersection).

<u>Corollary 4.2</u>. Let F be IHR, $F(0) = 0$, and let F have first and second moments $\mu_1 = 1$, and $\mu_2 \neq 1$. Then

$$1 - F(t) \geq \inf_{G \in \mathcal{B}_3} [1 - G(t)] = \inf_{T \geq T_1} e^{-a(t-\Delta)}, \quad t < 1 \qquad (4.11)$$

where a and Δ are determined by (4.5) and (4.6) as functions of T;

$$1 - F(t) \geq e^{-1}, \qquad t = 1 ,$$

$$1 - F(t) \geq \inf_{G \in \mathcal{B}_4} [1 - G(t)] = \inf_{T_0 \leq T \leq t} e^{-a_1 T - a_2 (t-T)}, \ 1 < t < T_1 , \ (4.12)$$

where a_1 and a_2 are determined by (4.8) and (4.9) as functions of T;

$$1 - F(t) \geq 0, \qquad t \geq T_1 .$$

The bounds are sharp.

The restriction $\mu_2 \neq 1$ is required in order that $1 - F(t) \geq e^{-1}$, $t = 1$; otherwise, $1 - F(t)$ must be replaced by $1 - F(t -)$.

<u>Proof</u>. For $T_0 < T < T_1$, and $x \leq T$, $1 - G_T(x) \geq 1 - G_{T_1}(x)$, since otherwise G_T and G_{T_1}, cannot cross twice. If $T < 1$, then since $1 - G_T(x)$ and $1 - G_{T_0}(x)$ must cross twice, $1 - G_T(T) \geq 1 - G_{T_0}(T)$. This together with $1 - G_T(1) \geq e^{-1} = 1 - G_{T_0}(1)$ (Theorem 3.8) implies $1 - G_T(x) \geq 1 - G_{T_0}(x)$, $T \leq x \leq 1$. But $G_{T_0} = G_\infty$, and thus (4.11) follows from (4.10).

If $T_1 < T < \infty$, then G_T and $G_{T_0} = G_\infty$ can cross only once in $(0,T)$. Since $1 - G_T(1) \geq 1 - G_{T_0}(1) = e^{-1}$, and since $1 - G_T(T_0) < 1 - G_{T_0}(T_0) = 1$, this crossing must occur in $(T_0, 1]$. Hence $1 - G_T(x) > 1 - G_{T_0}(x)$ for $1 < x < t$, and we conclude from (4.10) that $1 - F(t) \geq \inf_{\mathcal{B}_4} [1 - G(t)]$, $T_1 \geq t > 1$. The remainder of (4.12) follows from the fact that for $x < T < T_1$, $1 - G_T(x) > 1 - G_{T_1}(x)$ (otherwise G_T and G_{T_1} cannot cross twice).||

<u>Theorem 4.3.</u> Let F be IHR, $F(0) = 0$, and let F have first and second moments $\mu_1 = 1$ and μ_2. Then

$$1 - F(t) \leq 1 \quad , \qquad 0 \leq t \leq T_0 = 1 - (\mu_2 - 1)^{1/2}; \tag{4.13}$$

$$1 - F(t) \leq e^{-a_1 t}, \qquad T_0 < t \leq T_1, \tag{4.14}$$

where a_1 is determined by (4.8) and (4.9) with $T = t$;

$$1 - F(t) \leq e^{-a(t-\Delta)}, \qquad t \geq T_1, \tag{4.15}$$

where a and Δ are determined by (4.5) and (4.6) with $T = t$. These bounds are sharp.

<u>Proof</u>. Let us first assume that (4.8), (4.9) and (4.5)(4.6) have the required solutions. It is easily verfied from (4.5) and (4.6) that $\lim_{T \to \infty} \Delta = 1 - (\mu_2 - 1)^{1/2}$, and sharpness of (4.13) follows. Let $T_0 < t \leq T_1$ and suppose $1 - F(t) > {}^{-a_1 t}$; then $1 - F(x) > {}^{-a_1 x}$, $0 < x \leq t$.

If $T_1 < T < \infty$, then G_T and $G_{T_0} = G_\infty$ can cross only once in $(0,T)$. Since $1 - G_T(1) \geq 1 - G_{T_0}(1) = e^{-1}$, and since $1 - G_T(T_0) < 1 - G_{T_0}(T_0) = 1$, this crossing must occur in $(T_0, 1]$. Hence $1 - G_T(x) > 1 - G_{T_0}(x)$ for $1 < x < t$, and we conclude from (4.10) that $1 - F(t) \geq \inf\limits_{\mathcal{B}_4} [1 - G(t)]$, $T_1 \geq t > 1$. The remainder of (4.12) follows from the fact that for $x < T < T_1$, $1 - G_T(x) > 1 - G_{T_1}(x)$ (otherwise G_T and G_{T_1} cannot cross twice).||

<u>Theorem 4.3.</u> Let F be IHR, $F(0) = 0$, and let F have first and second moments $\mu_1 = 1$ and μ_2. Then

$$1 - F(t) \leq 1 \quad , \qquad 0 \leq t \leq T_0 = 1 - (\mu_2 - 1)^{1/2}; \tag{4.13}$$

$$1 - F(t) \leq e^{-a_1 t}, \qquad T_0 < t \leq T_1, \tag{4.14}$$

where a_1 is determined by (4.8) and (4.9) with $T = t$;

$$1 - F(t) \leq e^{-a(t-\Delta)}, \qquad t \geq T_1, \tag{4.15}$$

where a and Δ are determined by (4.5) and (4.6) with $T = t$. These bounds are sharp.

<u>Proof.</u> Let us first assume that (4.8), (4.9) and (4.5)(4.6) have the required solutions. It is easily verfied from (4.5) and (4.6) that $\lim\limits_{T\to\infty} \Delta = 1 - (\mu_2 - 1)^{1/2}$, and sharpness of (4.13) follows. Let $T_0 < t \leq T_1$ and suppose $1 - F(t) > e^{-a_1 t}$; then $1 - F(x) > e^{-a_1 x}$, $0 < x \leq t$.

Since F and G_t cross at least twice, this would force $1 - F(t)$ and $e^{-a_1 t - a_2(x-t)}$ to intersect three times which is impossible. If $t \geq T_1$, then $1 - F(t) > 1 - G_t(t)$ together with the fact that F and G_t cross at least twice would force $F(x)$ and $e^{-a(x-\Delta)}$ to cross three times and again we obtain a contradiction.

Theorem 4.3 also follows as a corollary of Theorem 4.1, since from Theorem 4.1, we need only show that $1 - G_t(t) \geq 1 - G_s(t)$ for all $s \neq t$; but this follows from the fact that G_t and G_s must cross twice.

To complete the proof of Theorem 4.3, it is necessary to show that (4.8), (4.9) and (4.5), (4.6) have the required solutions. This proof is given in

<u>Lemma 4.4</u>. For every T, $T > T_1$, there is a unique solution of (4.5) and (4.6). For every T, $T_0 \leq T \leq T_1$, there is a unique solution of (4.8), and (4.9). Furthermore, these solutions are continuous in T.

<u>Proof</u>. Consider first the case that $T > T_1$; fix $T > T_1$, $\Delta \in [0,1]$, and let

$$\alpha(a,T,\Delta) = a^{-1}(1 - e^{-a(T-\Delta)}) + \Delta - 1.$$

Then $\lim_{a\to\infty} \alpha(a,T,\Delta) = \Delta - 1$, $\lim_{a\to 0} \alpha(a,T,\Delta) = T - 1 \geq 0$ $(T_1 \geq 1)$ and $\frac{\partial\alpha(a,T,\Delta)}{\partial a} \leq 0$ for all a. Therefore $\alpha(a,T,\Delta) = 0$, i.e. (4.5), has a unique solution $a(T.\Delta)$ for each fixed Δ and T; furthermore $\alpha(a,T,\Delta) \underset{(>)}{\leq} 0$ for $a \underset{(<)}{\geq} a(T,\Delta)$. Let $\delta > 0$. Then $\alpha(a(T,\Delta) - \delta,T,\Delta) > 0$, $\alpha(a(T,\Delta) + \delta,T,\Delta) < 0$. By continuity of α, there exists $\epsilon_1 > 0$, $\epsilon_2 > 0$ (possibly depending on a,δ,T and Δ)

such that $|T - T'| < \epsilon_1, |\Delta - \Delta'| < \epsilon_2$ implies $\alpha(a(T,\Delta) - \delta, T', \Delta') > 0$, $\alpha(a(T,\Delta) + \delta, T', \Delta') < 0$. Hence there exists $a(T',\Delta')$, $a(T,\Delta) - \delta < a(T',\Delta') < a(T,\Delta) + \delta$, such that $\alpha(a(T',\Delta'),T',\Delta') = 0$. This proves that $a(T,\Delta)$ is continuous in T and Δ.

Let

$$K(\Delta,T) = \Delta^2 - 2a^{-2}(aT + 1)e^{-a(T-\Delta)} + 2a^{-2}(a\Delta + 1)$$

where $a = a(T,\Delta)$ is determined by (4.5). We want to show that $K(\Delta,T) = \mu_2$, i.e. (4.6), has a unique solution $\Delta(T)$ continuous in T. If $\Delta = 0$, (4.5) implies $e^{-aT} = 1 - a$, so that $K(0,T) = 2a^{-1}(1 - Te^{-aT})$, and

$$\partial K(0,T) / \partial T = 2a^{-2}[(1 - T)\partial a/\partial T - a(1 - a)] > 0$$

where $\partial a/\partial T = ae^{-aT}(1 - Te^{-aT})^{-1} = a(1 - a)(1 - Te^{-aT})^{-1}$ if

$$(1 - T)a(1 - a)(1 - Te^{-aT})^{-1} \geq a(1 - a),$$

which is clear if $0 \leq a \leq 1$. But this follows from $e^{-aT} = 1 - a$ and $T \geq 1$. Therefore $T \geq T_1$ implies

$$K(0,T) \geq K(0,T_1) = \mu_2 \geq 1 = \lim_{\Delta\to 1} K(\Delta,T).$$

This implies that $K(\Delta,T) = \mu_2$ has a solution $\Delta(T)$. Uniqueness of $\Delta(T)$ follows from the fact that for given T, there is at most one element of $\mathcal{B}_3$; two distributions in $\mathcal{B}_3$ are identical, or cross exactly twice, and the latter is impossible if they correspond to the same T.

Continuity of $\Delta(T)$ follows in the same manner as continuity of $a(T,\Delta)$. This completes the proof of Lemma 4.4 in case $T > T_1$.

Let $T_0 \leq T \leq T_1$. Solving (4.8) for a_2 as a function of a_1 and T, we obtain

$$a_2(a_1,T) = a_1 e^{-a_1 T}(a_1 - 1 + e^{-a_1 T})^{-1} ,$$

and substitution in (4.9) yields

$$h(a_1,T) = \{e^{-a_1 T}[1 + (2 + a_1 T)(a_1 - 1)] + (a_1 - 1)^2\}/a_1{}^2 e^{-a_1 T} = \mu_2/2 \quad .$$

It is easily verified that $h(1,T) = 1$ and $\lim_{a_1 \to 0} h(a_1,T) = (1 + (1 - T)^2)/2$. Now $T \geq T_0 = 1 - (\mu_2 - 1)^{1/2}$ implies $(1 + (1 - T)^2)/2 \leq \mu_2/2 \leq 1$. Since h is continuous, there exists $a_1 = a_1(T)$ satisfying $h(a_1,T) = \mu_2/2$. Furthermore, by arguments previously used, it can be shown that $a_1(T)$ is unique and continuous. ||

If F is IHR, there can be at most two crossings of $1 - F$ and an exponential. Furthermore, the crossing points must be well defined, since if $1 - F(x)$ and ce^{-bx} coincide for all x in some internal, then $1 - F(x) \leq ce^{-bx}$ for all x, and there can be no crossing. This is a simple consequence of the log concavity of $1 - F$.

Proof of Theorem 4.1. Let $F \in \mathcal{F} - (\mathcal{B}_3 \cup \mathcal{B}_4)$. For $T \geq T_1$, let $r(T)$ be the point in (Δ,T) that $1 - F$ crosses $1 - G_T$ from below if such a crossing exists ; otherwise, let $r(T) = \Delta$. Let $s(T)$ be the crossing in (Δ,T) from above of $1 - G_T$ by $1 - F$ if such a crossing exists; otherwise, let $s(T) = T$. Note that $r(T) \leq s(T)$.

For $T_0 \leq T \leq T_1$, let $u(T)$ be the crossing in (T,∞) from

below of $1 - G_T$ by $1 - F$; $u(T)$ always exists. Let $v(T)$ be the crossing in (T, ∞) from above of $1 - G_T$ by $1 - F$ if such a crossing exits; otherwise let $v(T)$ be the right-hand endpoint of the support of F.

In order to show that r, s, u, and v are continuous in the interior of their range suppose that F and G_T cross at $x = x_0$ (in case $T \geq T_1$, let $x_0 \neq T$). Choose $\epsilon > 0$ sufficiently small that $[G_T(x_0 - \epsilon) - F(x_0 - \epsilon)][G_T(x_0 + \epsilon) - F(x_0 + \epsilon)] < 0$ (and $x_0 + \epsilon < T$ when $T \geq T_1$). By Lemma 4.4, $G_T(x)$ is continuous in T for all x ($x < T$ in case $T \geq T_1$). Hence there exists $\delta > 0$ such that $|T' - T| < \delta$ implies

$[G_{T'}(x_0 - \epsilon) - F(x_0 - \epsilon)][G_{T'}(x_0 + \epsilon) - F(x_0 + \epsilon)] < 0$. This means that $G_{T'}$ and F cross in the internal $(x_0 - \epsilon, x_0 + \epsilon)$.

To show that for all $x < T_1$, there exists T such that F and G_T cross at x, it suffices to show that $\lim_{T \uparrow T_1} r(T) = 0$, $\lim_{T \to \infty} r(T) = u(T_0)$, and $\lim_{T \to T_1} u(T) = T_1$. The second two limits are clear from the definitions. Proof that $\lim_{T \uparrow T_1} r(T) = 0$ is similar to the proof of continuity.

To show that for all $x \geq T_1$, there exists T such that F and G_T cross at x, we note that $s(T_1) < T_1$, $\lim_{T \to \infty} s(T) = v(T_0)$, $\lim_{T \to T_1} v(T) =$ right-hand endpoint of the support of F.||

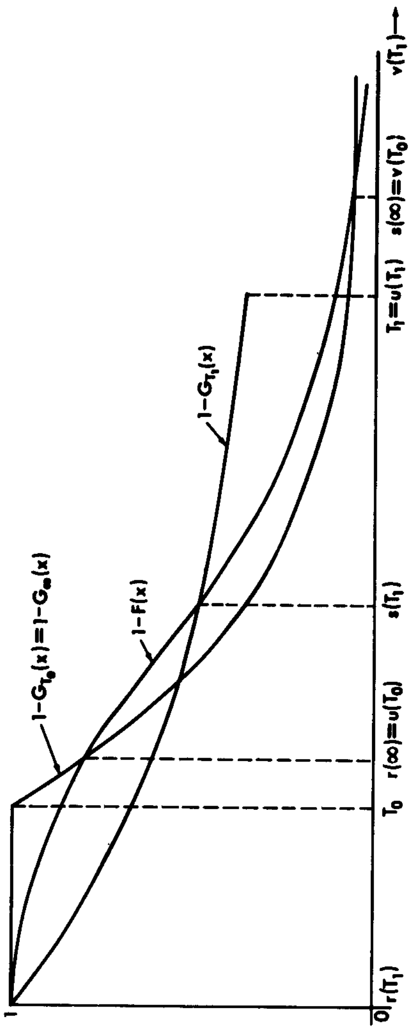

Figure 4.3

4.2. BOUNDS FOR $1 - F$ WHEN F IS DHR.

Let

$$1 - G_{T;\alpha}(x) = \begin{cases} \alpha e^{-a_1 x}, & 0 \le x \le T \\ \alpha e^{-a_2 x + (a_2 - a_1)T}, & x \ge T, \end{cases} \qquad 0 \le T \le \infty \tag{4.16}$$

where

$$\alpha^{-1} = a_1^{-1}(1 - e^{-a_1 T}) + a_2^{-1} e^{-a_1 T}, \tag{4.17}$$

$$\mu_2 / 2\alpha = a_1^{-2}[1 - (a_1 T + 1)e^{-a_1 T}] + a_2^{-2}(a_2 T + 1)e^{-a_1 T}. \tag{4.18}$$

Following the proof of Lemma 4.4, we conclude that for every $T \ge 0$ and every α, $2\mu_2^{-1} \le \alpha \le 1$, there exists a_1 and a_2 satisfying (4.17) and (4.18). Note that

$$1 - G_{\infty;2\mu_2^{-1}}(x) = 2\mu_2^{-1} e^{-2x/\mu_2}, \qquad x \ge 0.$$

Equations(4.17) and (4.18) insure that $G_{T;\alpha}$ has the first moment $\mu_1 = 1$ and second moment μ_2. Furthermore, $G_{T;\alpha}$ is DHR if $\mu_2 \ge 2$.

Theorem 4.5. If F is DHR, $F(0-) = 0$ and if F has first and second moments $\mu_2 = 1$ and μ_2, then

$$1 - F(t) \ge \begin{cases} 2\mu_2^{-1}, & t = 0 \\ e^{-a_1 t}, & t > 0, \end{cases} \tag{4.19}$$

where a_1 is determined by (4.17) and (4.18) with $\alpha = 1$ and $T = t$. The bound is sharp.

Note that since F is DHR, $\mu_2 \geq 2$.

<u>Proof</u>. Since F and $G_{\infty;2\mu_2^{-1}}$ have the same first two moments, they cross at least twice. Since F is DHR there are exactly two crossings, and the first crossing of $1 - G_{\infty;2\mu_2^{-1}}$ by $1 - F$ must be from above. Hence $1 - F(0) \geq 2\mu_2^{-1}$. Now let $t > 0$ and suppose that $1 - F(t) < 1 - G_{t;1}(t)$. Then since $1 - F(0) \leq 1 - G_{t;1}(0)$, F and $G_{t;1}$ can cross at most once in $[0,t]$, and it follows from $1 - F(t) < 1 - G_{t;1}(t)$ that there are no crossings in $[0,t]$. Since $1 - F(t) < 1 - G_{t;1}(t)$, there can be at most one crossing of F and $G_{t;1}$ in (t,∞). Hence F and $G_{t;1}$ cross at most once in $[0,\infty)$, contradicting the assumption that they have the same first two moments.||

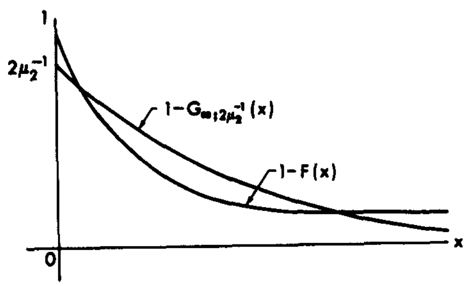

Figure 4.4

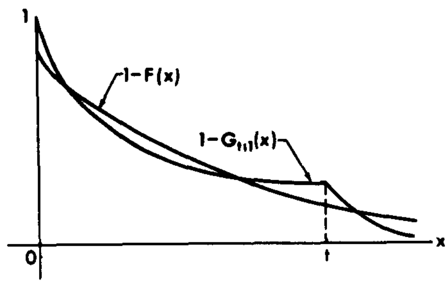

Figure 4.5

<u>Theorem 4.6</u>. If F is DHR, $F(0-)=0$ and F has first and second moments $\mu_1 = 1$ and μ_2, then

$$1 - F(t) \le \begin{cases} e^{-t}, & 0 \le t \le 1 \\ (te)^{-1}, & 1 \le t \le \mu_2/2 \\ 2\mu_2^{-1} e^{-2t/\mu_2}, & \mu_2/2 \le t \le \mu_2 \\ \sup\limits_{0<T<t} 1 - G_{T;1}(t), & t > \mu_2 . \end{cases} \tag{4.20}$$

These bounds are sharp.

<u>Proof</u>. Recall from (3.13) that

$$1 - F(t) \le \begin{cases} e^{-t}, & t \le 1 \\ (te)^{-1}, & t \ge 1. \end{cases}$$

We wish to show these bounds are sharp for $t \le \mu_2/2$. Let a_1, a_2 and α be determined by (4.17) and (4.18), and assume that $\mu_2 > 2$ (so that F is not exponential). Then since F is DHR, $a_1 > 1 > a_2$. Hence by (4.17), $\lim\limits_{T\to\infty} a_2 =$ $\lim\limits_{T\to\infty} a_1[(a_1\alpha^{-1} - 1)e^{a_1T} + 1]^{-1} = 0$. By (4.18), $\lim\limits_{T\to\infty} a_2^{-1}e^{-a_1T} < \mu_2/2\alpha < \infty$, so that $\lim\limits_{T\to\infty} a_2^{-1}e^{-a_1T} = 0$. Hence from (4.7) and $a_1 > 1$ we conclude $\lim\limits_{T\to\infty} a_1 = \alpha$. This means

that $\lim_{T\to\infty} 1 - G_{T;\alpha}(x) = \alpha e^{-\alpha x}$. Since $\lim_{T\to\infty} 1 - G_{T;1}(t) = e^{-t}$, (4.20) is sharp for $t \leq 1$. Since $\lim_{T\to\infty} 1 - G_{T;1/t}(t) = (te)^{-1}$, (4.20) is sharp for $1 \leq t \leq \mu_2/2$. Note that for $t = \mu_2/2$, equality is attained by the distribution $1 - G_{\infty;2\mu_2^{-1}}$.

Next, recall from (3.13) that

$$1 - F(t) \leq 2e^{-2}\mu_2/t^2, \quad t \geq 2\mu_2^{1/2};$$

this proves (4.20) for $t = \mu_2$. Equality is attained in (4.20) for $t = \mu_2$ again by the distribution $1 - G_{\infty;2\mu_2^{-1}}$.

We have shown that $1 - F(t) \leq 1 - G_{\infty;2\mu_2^{-1}}(t)$ for $t = \mu_2/2$ and $t = \mu_2$. Since F is DHR, this implies $1 - F(t) \leq 1 - G_{\infty;2\mu_2^{-1}}(t)$ for all t in $[\mu_2/2, \mu_2]$, so that (4.20) holds for $\mu_2/2 \leq t \leq \mu_2$.

Finally, we consider the case $t > \mu_2$. Since $1 - F(0) \leq 1 - G_{T;1}(0)$, there is at most one crossing of $1 - G_{T;1}$ by $1 - F$ in $(0,T]$, and hence there is a first crossing $u(T)$ to the right of T. Since $1 - G_{T;1}(T) \leq 1 - F(T)$ by (4.19), this crossing is from above. Since $u(T) > T$, $\lim_{T\to\infty} u(T) = \infty$. Since $\lim_{T\to 0} 1 - G_{T;1}(x) = 1 - G_{\infty;2\mu_2^{-1}}(x)$, $\lim_{T\to 0} 1 - G_{T;1}(\mu_2) = 2\mu_2^{-1}e^{-2} \geq 1 - F(\mu_2)$, and hence $\lim_{T\to 0} u(T) \leq \mu_2$. By arguments similar to those of section 4.1, it follows that $u(T)$ is continuous in T, so that for every $t > \mu_2$, there exists $T < t$

such that $u(T) = t$, that is, $1 - G_{T;1}(t) = 1 - F(t)$. ||

We remark that the bounds of (4.20) are sharp with the additional assumption that $F(0) = 0$. However, it may be that the bound can only be approximated, and equality is unattainable.

Remarks on Generalizations. We conjecture that in case the first n moments are given that the family of extremal distributions again consists of piecewise exponentials with n pieces, and the possibility of truncation on the right. Indeed, it is possible to show that such distributions are extremal by arguments essentially the same as used to prove Theorems 4.3 and 4.5. However, one requires a generalization of Lemma 4.4 (or its DHR analog). Here, one would like to know the solution to the moment problem for IHR (DHR) distributions.

5. UPPER BOUNDS FOR $1 - F$ WHEN f IS PF_2.

In this section, we obtain a sharp upper bound for $1 - F(t)$, given a single expectation $\int_0^\infty \zeta(x)dF(x) = v$ (ζ monotone), when $F(0) = 0$ and F has a density f that is a Pólya frequency function of order 2 (PF_2). Briefly, f is PF_2 if $\log f(x)$ is concave on the support of F, an interval (see Schoenberg (1951) for a precise definition). The condition that f is PF_2 implies that F is IHR (Barlow, Marshall, Proschan (1963)), so that the result here is a sharpening of inequality (3.5).

Under the condition that f is PF_2, no sharpening of (3.1) is possible, since the extremal distributions of (3.1) are exponential, and therefore have PF_2 (indeed, PF_∞) densities.

Let

$$G_m(x;b) = \begin{cases} (1 - e^{-bx})/(1 - e^{-bm}), & 0 \le x \le m \\ 1, & x > m, \end{cases}$$

for $m > 0$ and $b \neq 0$; let $G_m(x;0) = \lim_{b \to 0} G_m(x;b)$. This distribution has a density

$$g_m(x;b) = \begin{cases} be^{-bx}/(1 - e^{-bm}), & 0 \le x \le m \\ 0, & \text{elsewhere,} \end{cases}$$

which is obtained by truncating an exponential density. Hence g_m is PF_2.

Theorem 5.1. Let f be a PF_2 density such that $f(x) = 0$ for $x < 0$. Let ζ be a function continuous and strictly increasing on $[0,\infty)$ such that $\int_0^\infty \zeta(x)dF(x) = v$ exists finitely. Then for each $m > \zeta^{-1}(v)$,

there exists a unique b_m satisfying

$$\int_0^\infty \zeta(x)dG_m(x;b_m) = \nu. \tag{5.1}$$

Furthermore, for all $t > 0$,

$$1 - F(t) \leq \begin{cases} 1, & t < \zeta^{-1}(\nu) \\ \sup_{m \geq t}[1 - G_m(t;b_m)], & t \geq \zeta^{-1}(\nu). \end{cases} \tag{5.2}$$

In the case that $\zeta(x) \equiv x$, this bound has been computed numerically, and is graphed in Figure 6.1.

Before proving Theorem 5.1, we prove some useful lemmas.

Lemma 5.2. $\int_0^\infty \zeta(x)dG_m(x;b) \equiv \Phi(b,m)$ is continuous in b for fixed m and continuous in m for fixed b.

Proof. Since $\lim_{b \to b^*} G_m(x;b) = G_m(x,b^*)$, and $\lim_{m \to m^*} G_m(x;b) = G_{m^*}(x;b)$ for all $m^* > 0$ and all b^*, the theorem follows from the Helly-Bray lemma (Loève, 1960, p. 182). ||

Lemma 5.3. For all $m > 0$ and $\nu \varepsilon [\zeta(0),\zeta(m)]$ there exists a unique b_m satisfying (5.1).

Proof. We first show that $G_m(x;b)$ is strictly increasing in b for each $x < m$. If $b \neq 0$, $\partial G_m(x;b)/\partial b > 0$ if and only if $\varphi(x) > \varphi(m)$ where $\varphi(z) = ze^{-bz}/(1 - e^{-bz})$. But $\varphi'(z) = e^{-bz}(1 - bz - e^{-bz})/(1 - e^{-bz})^2 < 0$ for all $bz \neq 0$. Hence for $x < m$ and $b \neq 0$, $\varphi(x) > \varphi(m)$. If $b = 0$, then $\partial G_m(x;b)/\partial b\big|_{b=0} = x(m - x)/2m > 0$ for $x < m$. Thus $G_m(x;b)$ is

strictly increasing in b for each $x < m$, and hence $\Phi(b,m) \equiv \int\zeta(x)dG_m(x,b)$ is strictly <u>decreasing</u> in b (since ζ is increasing). Since $\Phi(b,m)$ is continuous in b by Lemma 5.2, it remains only to show that $\lim_{b\to\infty} \int\zeta(x)dG_m(x;b) = \zeta(0)$ and $\lim_{b\to-\infty} \int\zeta(x)dG_m(x;b) = \zeta(m)$. But this follows by the Helly-Bray lemma, since

$$\lim_{b\to\infty} G_m(x;b) = \begin{cases} 0, & x \leq 0 \\ 1, & x > 0, \end{cases} \quad \text{and} \quad \lim_{b\to-\infty} G_m(x;b) = \begin{cases} 0, & x < m \\ 1, & x \geq m. \end{cases} \quad ||$$

For convenience, we introduce the notation

$$g_m(x) \equiv g_m(x;b_m).$$

<u>Lemma 5.4</u>. $g_m(t)$ is continuous in $m \geq t$.

<u>Proof</u>. It is sufficient to show that b_m is continuous in m, where b_m is determined by (5.1). Let $\varepsilon > 0$ and fix m. Since $\Phi(b,m)$ is decreasing in b (see the proof of Lemma 5.3), and since Φ is continuous in b (Lemma 5.2), there exists $\eta > 0$ such that $\Phi(b_m + \varepsilon,m) > \nu - \eta$ and $\Phi(b_m - \varepsilon,m) < \nu + \eta$. Now since Φ is continuous in m, there exists $\delta > 0$ such that $|m - m'| < \delta$ implies $\Phi(b_m + \varepsilon,m') > \nu - 2\eta, \Phi(b_m - \varepsilon,m') < \nu + 2\eta$. Then by monotonicity and continuity of Φ, $b_{m'} \in (b_m - \varepsilon, b_m + \varepsilon)$. That is, $|b_m - b_{m'}| < \varepsilon$ whenever $|m - m'| < \delta$. ||

Suppose that for all m, $f \not\equiv g_m$. Then g_m crosses f exactly once from below; since $\log f(x)$ is concave and $\log g_m(x)$ is linear in $x \in [0,m)$, there is at most one such crossing (see Karlin, Proschan and Barlow (1961)). By (5.1), F and G_m must cross at least once (f and g_m must cross twice) so that there exists at least one such crossing. Denote the unique such crossing point by $x^*(m)$.

<u>Lemma 5.5</u>. $x^*(m)$ is continuous in m.

<u>Proof</u>. Fix m and let $x^*(m) = x^*$. Since g_m crosses f from below at x^*, there exists $\varepsilon > 0$ such that

$$f(x) - g_m(x) \begin{cases} > 0, & x^* - 2\varepsilon < x < x^* \\ < 0, & x^* < x < x^* + 2\varepsilon . \end{cases}$$

Let $2\eta = \min\{[f(x^* - \varepsilon) + g_m(x^* - \varepsilon)], [g_m(x^* + \varepsilon) - f(x^* + \varepsilon)]\}$. Since $g_m(x)$ is continuous in $m > x$ for fixed x, there exists $\delta > 0$ such that $|m - m'| < \delta$ implies $|g_m(x) - g_{m'}(x)| < \eta$ for $x = x^* \pm \varepsilon$. Then

$$|f(x^*+ \varepsilon) - g_{m'}(x^*+ \varepsilon)| \geq |f(x^*+ \varepsilon) - g_m(x^*+ \varepsilon)| - |g_m(x^*+ \varepsilon) - g_{m'}(x^*+ \varepsilon)| > \eta,$$

$$|f(x^*- \varepsilon) - g_{m'}(x^*- \varepsilon)| \geq |f(x^*- \varepsilon) - g_m(x^*- \varepsilon)| - |g_m(x^*- \varepsilon) - g_{m'}(x^*- \varepsilon)| > \eta.$$

By continuity (in x) of $f(x)$ and $g_{m'}(x)$, $x^*(m') \in (x^*(m) - \varepsilon, x^*(m) + \varepsilon)$; i.e., $|m - m'| < \delta$ implies $|x^*(m) - x^*(m')| < \varepsilon$. ||

<u>Proof of Theorem 5.1</u>. We suppose without loss of generality that $f \not\equiv g_m$ for all $m > 0$ and consider the case that $t \geq \zeta^{-1}(\nu)$. Since $\int \zeta(x)dF(x) = \nu$, it follows that $\zeta(0) \leq \nu$, and for $m \geq t \geq \zeta^{-1}(\nu)$, $\nu \leq \zeta(m)$. Thus b_m satisfying (5.1) exists uniquely by Lemma 5.3 for all $m \geq t$. Now assume that $t < x^*(\infty)$ (otherwise the theorem is obvious). Clearly $x^*(t) < t$. Hence by Lemma 5.5 there exists m_0 such that $x^*(m_0) = t$. Since g_{m_0} is logarithmically linear in $x < m_0$, f and g_{m_0} can cross at most twice in $(0,m_0)$. If there are two such crossings, then since g_{m_0} crosses f from below at t, the other crossing point x_1 satisfies $x_1 < t$. If there is only one such crossing (at t by choice of m_0), let $x_1 = 0$.

Then in either case,

$$f(x) \begin{cases} < g_{m_0}(x), & 0 < x < x_1 \text{ or } t < x < m_0 \\ \geq g_{m_0}(x), & x_1 < x < t \text{ or } x > m_0. \end{cases} \tag{5.3}$$

Let $\ell(x) = \alpha + \beta\zeta(x)$, where $\beta = [\zeta(m_0) - \zeta(x_1)]^{-1}$ and $\alpha = -\zeta(x_1)\beta$. Then $\chi_{[t,\infty)}(x) - \ell(x)$ changes sign with $f(x) - g_{m_0}(x)$, and consequently

$$[\chi_{[t,\infty)}(x) - \ell(x)][f(x) - g_{m_0}(x)] \leq 0. \tag{5.4}$$

Integration on x from 0 to ∞ yields

$$\int_t^\infty f(x)dx \leq \int_t^\infty g_{m_0}(x)dx. \quad ||$$

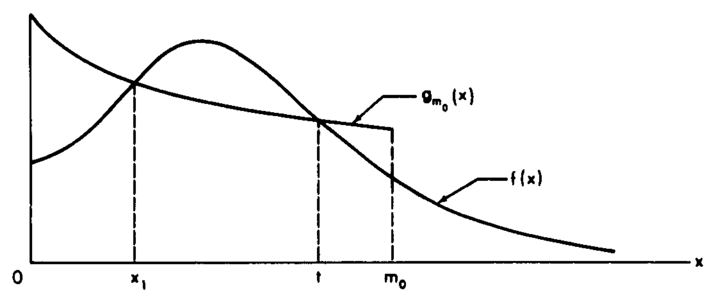

Figure 5.1

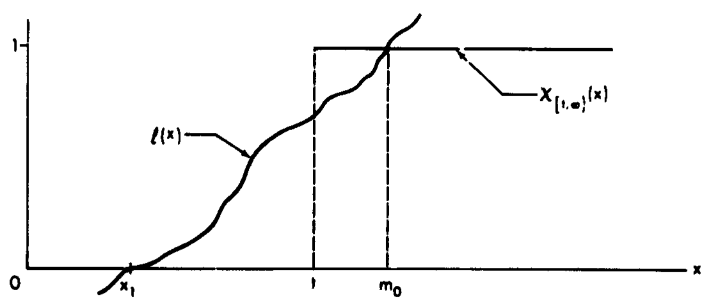

Figure 5.2

Theorem 5.5. Inequality (5.2) is sharp.

Proof. In case $\nu \leq \zeta(t)$ the theorem is obvious; in case $\nu > \zeta(t)$, equality is attained by the distribution degenerate at ν. This degenerate distribution can occur in many ways as a limit of distributions with PF_2 densities. ||

Corollary 5.6. Let f be a PF_2 density such that $f(x) = 0$, $x < 0$, and such that $\int_0^t f(x)dx = p$. If ζ is a function continuous and strictly increasing on $[0,\infty)$, then

$$\int_0^\infty \zeta(x)f(x)dx \geq \inf_{m>t} \int_0^\infty \zeta(x)g_m(x;c_m)dx, \tag{5.5}$$

where for each $m > t$, c_m is uniquely determined by

$$\int_0^t g_m(x;c_m)dx = p. \tag{5.6}$$

Proof. $G_m(t;c)$ is strictly increasing in c (see the proof of Lemma 5.3), $\lim_{c\to\infty} G_m(t;c) = 1$ and for $t < m$, $\lim_{c\to-\infty} G_m(t;c) = 0$. Hence (5.6) has a unique solution c_m for each $m > t$ and $p\varepsilon(0,1)$.

Consider now the case that $\zeta(t) \geq \nu = \int_0^\infty \zeta(x)f(x)dx$. Let m_0 be as defined in the proof of Theorem 5.1. Then by (5.4), $G_{m_0}(t;b_{m_0}) \leq F(t)$ $= G_{m_0}(t;c_{m_0})$ so that $G_{m_0}(x;b_{m_0}) \leq G_{m_0}(x;c_{m_0})$ for all x. This together with monotonicity of ζ yields

$$\int_0^\infty \zeta(x)f(x)dx = \int_0^\infty \zeta(x)g_{m_0}(x;b_{m_0})dx \geq \int_0^\infty \zeta(x)g_{m_0}(x;c_{m_0})dx$$

$$\geq \inf_{m>t} \int_0^\infty g_m(x;c_m)dx.$$

Next suppose that $\zeta(t) < \nu$. Since, for fixed c, $\lim_{m \downarrow t} G_m(t;c) = 1$, it follows that $\lim_{m \downarrow t} c_m = -\infty$. Since ζ is continuous at t, $\lim_{m \downarrow t} \int_0^\infty \zeta(x) g_m(x;c_m)dx = \zeta(t) < \int_0^\infty \zeta(x) f(x)dx$. ||

Theorem 5.1 remains true if ζ is strictly decreasing rather than increasing. In this case, the statements of the lemmas remain unchanged and the proofs require only minor modifications. Inequality (5.4) is replaced by

$$[\chi_{[0,t]}(x) - \ell(x)][f(x) - g_{m_0}(x)] \geq 0,$$

where $\ell(x) = \alpha + \beta\zeta(x)$ and $\beta = [\zeta(x_1) - \zeta(m_0)]^{-1}$, $\alpha = -\zeta(m_0)\beta$. If ζ is decreasing rather than increasing, the direction of inequality (5.5) is reversed, and the infimum is replaced by supremum.

For Theorem 5.1, the continuity of ζ was used only for the applications of the Helly-Bray lemma in Lemmas 5.2 and 5.3. This condition can be relaxed, as can the condition that ζ be <u>strictly</u> monotone. In particular, (5.2) holds if for some $s > t$, $\zeta(x) = \chi_{[s,\infty)}(x)$ (i.e., if ν is a percentile).

We remark that for $t \geq \nu$,

$$f(t) \geq g_t(t). \tag{5.7}$$

This inequality follows from arguments similar to those advanced in the discussion preceding Lemma 5.5. Further bounds for densities will appear in a forthcoming paper by the authors.

Note that g_m is <u>not</u> PF_3. This means that in case f is PF_3, inequality (5.2) is not sharp, but can be improved.

6. SOME NUMERICAL COMPARISONS.

Extensive tables for various bounds of Sections 3, 4, and 5 that have no explicit expressions are given in Barlow and Marshall (1963). We present here some numerical results in the form of graphs, and make comparisons with several other bounds, which are listed below:

(1) If $F(0-) = 0$, F is concave on $[0,\infty)$ (i.e., the density f is decreasing on $[0,\infty)$), and $\mu_1 = 1$, then an upper bound for $1 - F(t)$ due to Camp (1922) and Meidell (1922) is given by (2.7).

(2) If f is unimodal (more generally, if F is convex on $[0,m]$ and concave on $[m,\infty)$ for some unknown m), and $\mu_1 = 1$, then

$$1 - F(t) \leq \begin{cases} 1, & 0 \leq t \leq 1 \\ 2t^{-1} - 1, & 1 \leq t \leq 3/2 \\ 1/2t, & t \geq 3/2. \end{cases} \tag{6.1}$$

This bound follows from the general theory given by Mallows (1962) and was communicated to us by Prof. Mallows. Inequality (6.1) may be proved using an appropriate modification of the method illustrated by Example 2.2 assuming first that the location of the mode is known.

(3) In case $F(0-) = 0$, $\mu_1 = 1$ and μ_2 is also known, the following upper and lower bounds for $1 - F(t)$ are consequences of results given by Chebyshev (1874):

$$1 - F(t) \leq \begin{cases} 1, & 0 \leq t \leq 1 \\ t^{-1}, & 1 \leq t \leq \mu_2 \\ (\mu_2 - 1)/[\mu_2 - 1 + (t - 1)^2], & t \geq \mu_2; \end{cases} \tag{6.2}$$

$$1 - F(t) \geq \begin{cases} (1-t)^2/[(\mu_2 - 1) + (1-t)^2], & 0 \leq t \leq 1 \\ 0, & t \geq 1. \end{cases} \tag{6.3}$$

(4) By incorporating the hypothesis of (1) that F is concave on $[0,\infty)$, Chebyshev's results (6.2) and (6.3) have been improved by Royden (1953), as follows:

$$1 - F(t) \leq \begin{cases} 1 - t/2, & 0 \leq t \leq 1 \\ (2t)^{-1} & 1 \leq t \leq 3\mu_2/4 \\ 4(3\mu_2 - 2t)/9\mu_2^2, & 3\mu_2/4 \leq t \leq \mu_2 \\ \dfrac{3\mu_2 - 4}{4(3\alpha^2 - 4\alpha) + 3\mu_2} & \text{where } t = \dfrac{16\alpha^2(\alpha - 1)}{4(3\alpha^2 - 4\alpha) + 3\mu_2}, \ t \geq \mu_2; \end{cases} \tag{6.4}$$

$$1 - F(t) \geq \begin{cases} (2-t)^2/(3\mu_2 - 2t), & 0 \leq t \leq 2 \\ 0, & t > 2. \end{cases} \tag{6.5}$$

Assuming that $\mu_1 = 1$, the graphs of Figure 6.1 give upper bounds on $1 - F(t)$ in the cases of: general $F(1.1)$; unimodal $f(6.1)$; IHR $F(3.10)$; PF_2 $f(5.4)$. Recall that f is PF_2 implies both that F is IHR (Barlow, Marshall, and Proschan, 1963), and that f is unimodal (Schoenberg, 1951). However, IHR distributions need not have unimodal densities (Barlow, Marshall, and Proschan, 1963).

Figure 6.2 again gives upper bounds for $1 - F(t)$ with $\mu_1 = 1$. Here Markov's inequality (1.1) is given together with the improvements in case f is decreasing (2.7), and in case F is DHR (3.13). We recall that F is DHR implies that F is concave (f is decreasing).

Figures 6.3 (a,b, and c) show the upper and lower bounds of Chebyshev (6.2), (6.3) together with their IHR improvements given in Corollary 4.2 and Theorem 4.3. The striking improvement in the IHR case with $\mu_2 = 1.8$ is partially explained by the fact that if F is IHR with $\mu_1 = 1$ and $\mu_2 = 2$, then F is exponential.

Figure 6.4 for $\mu_1 = 1$, $\mu_2 = 3$ shows the sharp upper and lower bounds of Chebyshev ((6.2),(6.3)), their improvements in case f is decreasing (F is concave) on $[0,\infty)$, ((6.4),(6.5)), and their further improvements in case F is DHR, given in Theorems 4.5 and 4.6.

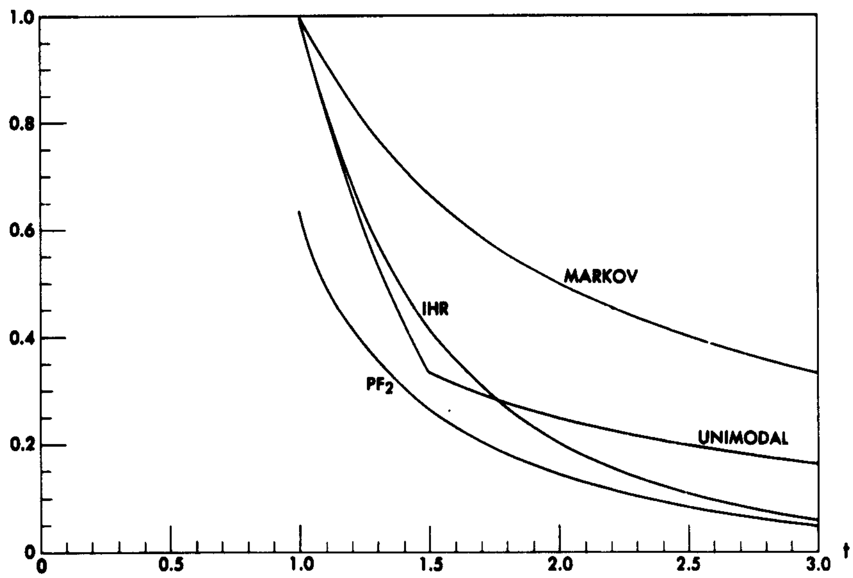

Figure 6.1 Upper bounds for $1 - F(t)$ $(\mu_1 = 1)$

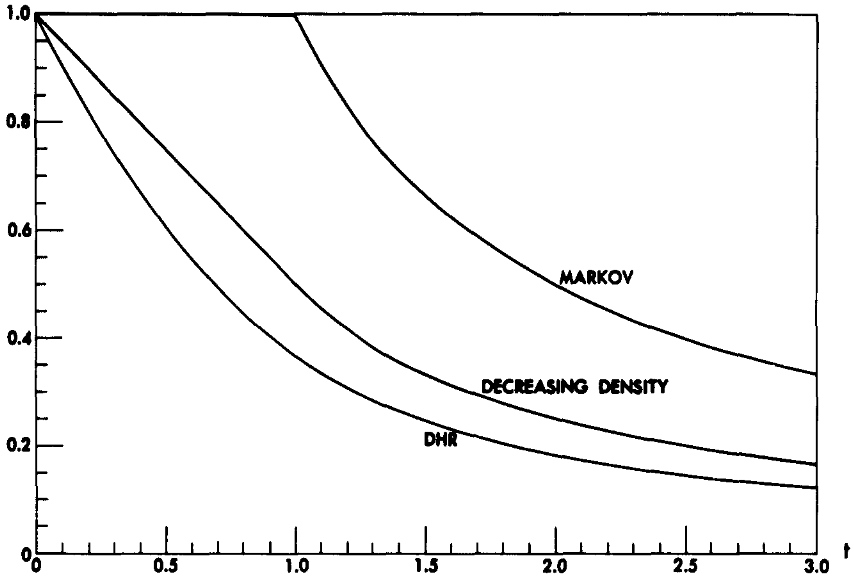

Figure 6.2. Upper bounds for $1 - F(t)$ $(\mu_1 = 1)$

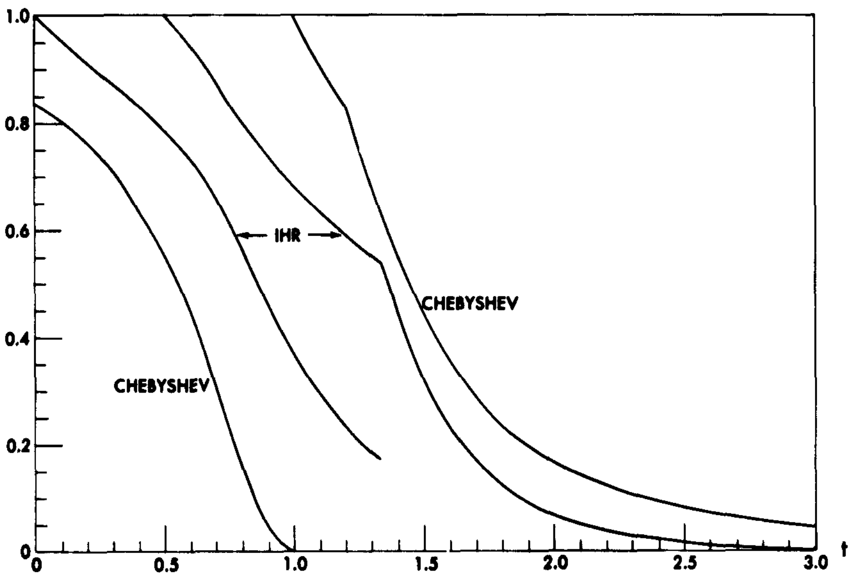

Figure 6.3a. Upper and lower bounds for $1 - F(t)$ $(\mu_1 = 1, \mu_2 = 1.2)$

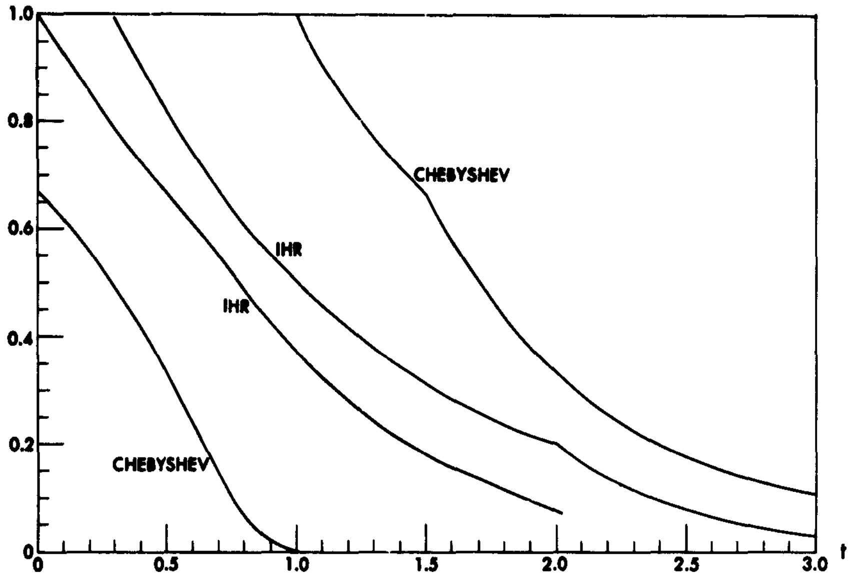

Figure 6.3b. Upper and lower bounds for $1 - F(t)$ ($\mu_1 = 1, \mu_2 = 1.5$)

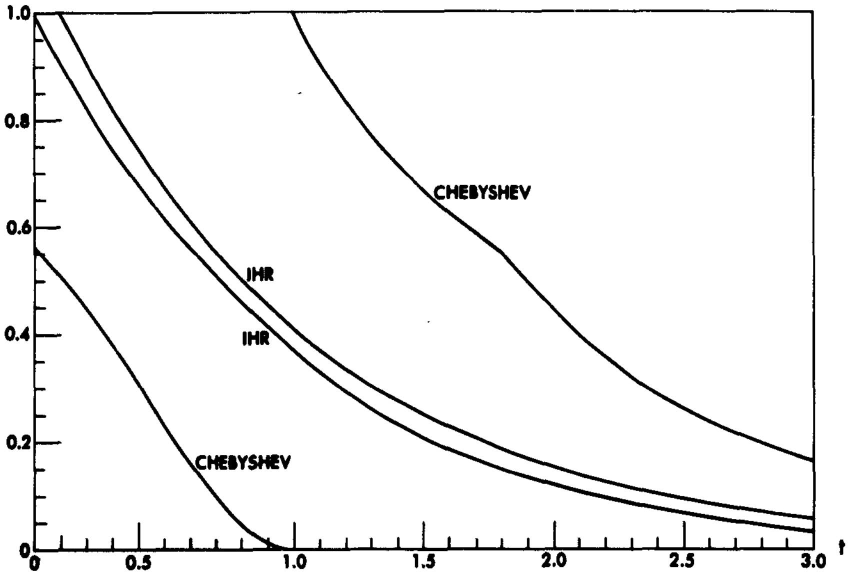

Figure 6.3c. Upper and lower bounds for $1 - F(t)$ ($\mu_1 = 1, \mu_2 = 1.8$)

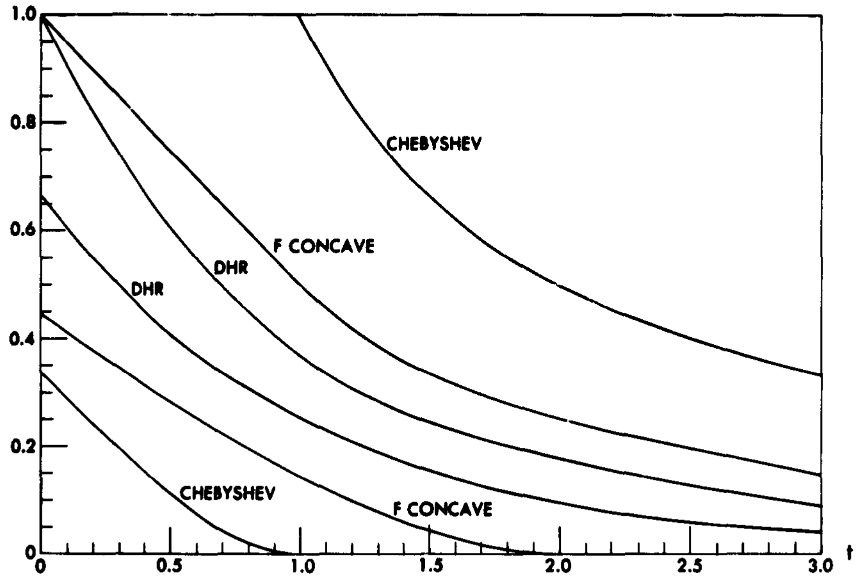

Figure 6.4 Upper and lower bounds for $1 - F(t)$ $(\mu_1 = 1, \mu_2 = 3)$

7. SOME REMARKS ON GENERALIZATIONS.

The arguments of this paper which depend on convexity properties of $\log[1 - F(x)]$ have been in several instances illustrated in Section 2 assuming convexity properties of F itself. This suggests that the two theories can be unified by appropriate generalizations, and in this section we indicate how this can be done.

A central role in the theory of distributions with monotone hazard rate is played by the exponential distribution. The simultaneous importance of the exponential function and the log function (which appears in the definition of IHR) suggests the following

Theorem 7.1. Let G be a distribution function with $G(0-) = 0$, suppose that the support of G is an interval, and let $\int_{0-}^{\infty} [1 - G(x)]dx = 1$. Then $H(x) = (1 - G)^{-1}(x)$ is defined for all x satisfying $0 < G(x) < 1$. If $H(1 - F(x))$ is convex, $F(0-) = 0$, and $t < \mu_1 = \int_{0-}^{\infty} [1 - F(x)]dx$, then

$$1 - F(t) \geq 1 - G(t/\mu_1). \tag{7.1}$$

The inequality is reversed if $H(1 - F(x))$ is concave.

Similar results can be obtained in case μ_1 is replaced by the expectation of an arbitrary increasing function. Inequality (7.1) can be proved using the method of Example 2.2; it is sharp, with equality attained by the distribution $G(x/\mu_1)$.

Inequality (7.1) is to be compared with (3.8), in which case $G(x) = 1 - e^{-x}$. Choosing $G(x) = x/2$, $0 \leq x \leq 2$, and assuming

$H(1 - F(x))$ is concave, one obtains the first bound of (2.7) with $r = 1$.

The direct proof given for (3.10) actually utilized only the condition that $x^{-1} \log[1 - F(x)] \geq t^{-1} \log[1 - F(t)]$, $x \leq t$, which is satisfied, e.g., by IHR distributions. Let $\psi_x(\cdot)$ be a strictly decreasing continuous function on $[0,1]$ (in particular, we may take $\psi_x(u) = x^{-1} \log u$), and suppose that $\varphi(z) = \int_0^t \psi_x^{-1}(z)dx$ is continuous. Let $\Psi(x) = \psi_x(1 - F(x))$.

<u>Theorem 7.2</u>. If $\int_{0-}^{\infty} xdF(x) = \mu_1 < \infty$, if $\Psi(x) \leq \Psi(t)$, $0 \leq x \leq t$, and if $\varphi(0) \geq \mu_1 \geq \varphi(\infty)$, then there exists a unique z_0 satisfying $\varphi(z_0) = \mu_1$. For z_0 so defined,

$$1 - F(t) \leq \psi_t^{-1}(z_0). \tag{7.2}$$

The proof of (7.2) is essentially the same as the direct proof given for (3.10).

If $\psi_0^{-1}(z_0) \leq 1$ and $\psi_x^{-1}(z_0)$ is decreasing in x, the distribution

$$1 - G(x) = \begin{cases} \psi_x^{-1}(z_0), & x \leq t \\ \\ 0, & x > t \end{cases}$$

attains equality in (7.2).

As previously indicated, (7.2) reduces to (3.10) with $r = 1$ in case $\psi_x(u) = - x^{-1} \log u$; the condition $\varphi(0) \geq \mu_1 \geq \varphi(\infty)$ is satisfied when $t \geq \mu_1$. If $\psi_x(u) = \psi(u)$ for all x, (7.2) reduces to (1.1) with

$r = 1$. With $\psi_x(u) = (1 - u)/x$, $\Psi(x) \leq \Psi(t)$ becomes $x^{-1}F(x) \leq t^{-1}F(t)$, $x \leq t$, which is true if F is convex in $x \leq t$, and (7.2) reduces to (2.4) with $r = 1$. Again the condition $\varphi(0) \geq \mu_1 \geq \varphi(\infty)$ is satisfied when $t \geq \mu_1$.

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